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STABILITY ANALYSIS OF AN ACTIVE VEHICLE SUSPENSION SYSTEM

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ABSTRACT: A car suspension system is widely applied as it provides a smooth ride and steering stability. With its usefulness, developments and creations of new suspensions are being continuously conducted. This results in drivers and passengers having a better experience when driving on rough roads. In this study, an active suspension is developed by analyzing a working process of a car suspension using the fundamental equations of constrained motion and Lyapunov Theory. MATLAB software is used to analyze the data and to examine the accuracy of the results. The results obtained from the analyzing process were found to be satisfactory. The research is conducted according to the objectives and limitations of the study which are successful in finding a mathematical model for the controller design of an active suspension system. Analyzing the functions of the controller, it was found that spring constant of choke, spring constant of tire, and damping constant affect the smoothness of a car suspension system.

Keywords: Stability; Suspension System; Fundamental Equation; Constrained Motion

1. INTRODUCTION

Nowadays, smooth driving is the main objective in developing modern cars and is the key to customer satisfaction. If cars show a good suspension response, the passengers' satisfaction would be higher with those particular cars. The currently favored type of suspension for further development is an Active Suspension [1-3]. This is due to its ability to constantly adjust forces applied to the suspension system by calculating suitable forces for each type of road at any period of time. In the study model, it is assumed that the suspension could apply forces to both sprung mass and unsprung mass, which are comparable to a car cabin and a car suspension, respectively. According to many studies related to the active suspension [4-5], it is easier to consider a dynamic model of a quarter car and to control the system with feedback controller [6-7]. With that in mind, if errors were to be found, the controller would try to return the system back to the equilibrium point. In this paper, the fundamental equations of constrained motion [8-11] are used to solve the dynamic equations of the quarter car mathematical model in order to verify the stability of the controller. In addition, Lyapunov Theory is applied to determine the stability constraint of the system. The linear constraint function in the form of acceleration is then received. Its condition helps diminish the energies in the system to zero or close to the equilibrium point, which meets the designed expectation. The verification of the effectiveness of the proposed controller is done by a numerical computation in MATLAB to find the displacement and velocity in each period of time of the sprung and unsprung masses of the quarter car. This is to prove that the system meets the stability's expectations and design objectives.

2. DYNAMICS

The fundamental equations of constrained motion are used to analyze the target system. The approach is divided into three steps [11-15].

2.1 Unconstrained System

First, Newton second law of motion is used to create a non-constrained system's equation of motion.

$$\sum F = ma. \tag{1}$$

Eq. (1) can be rearranged and written as

$$M\ddot{q} = Q. \tag{2}$$

where

M is a mass matrix (kg)

$\ddot{q}$ is a generalized acceleration ($\mathrm{m/s^2}$)

Q is a generalized force (N).

In pre-multiplied Eq. (2) with M^{-1}, we can find the unconstrained system's acceleration as

$$a := \ddot{q} = M^{-1}Q. \quad (3)$$

2.2 Constraint Equation

In this part, a constraint function (or a part which forces the system back to a desired stage) is defined for the system to be back to the equilibrium point:

$$\phi(q, \dot{q}, t) = 0. \quad (4)$$

After differentiating Eq. (4) and rewriting it in acceleration form [12], we get:

$$A(q, \dot{q}, t)\ddot{q} = b(q, \dot{q}, t) \quad (5)$$

where
A is a constraint matrix
b is a constant vector.

2.3 Constrained System

In order for the system to satisfy the constraint (5), based on the Newton second law of motion (Eq. (2)), we use forces from the feedback controller as a control force [13,14]:

$$M\ddot{q} = Q(q, \dot{q}, t) + Q^C(q, \dot{q}, t). \quad (6)$$

where
Q^C is the control force (N), which is computed as [9]:

$$Q^C = A^T(AM^{-1}A^T)^+(b - A\ddot{q}) \quad (7)$$

where + is the Moore-Penrose inverse matrix [9].

Then, Eq. (7) is substituted with Eq. (6) and the equation is pre-multiplied with M^{-1}. Finally, the dynamic equations are obtained in the form of acceleration:

$$\ddot{q} = a + M^{-1}A^T(AM^{-1}A^T)^+(b - Aa). \quad (8)$$

In order to verify the motions of the constrained dynamic system, Eq. (8) is integrated twice to compute the velocity and displacement of the system in any period of time. After that, the obtained values were used to calculate errors in satisfying the constraints. They were also used to have a guideline for adjusting the design parameters for the controller to achieve the desired goal of the constrained system.

3. METHOLOGY

3.1 Analyzing the Unconstrained System

In this work, a quarter car is modeled with an active suspension as seen in Fig.1.

In its free body diagram, a vertical movement of a two-degree of freedom system is considered. By the virtue of the unconstrained motion, the Newton second law of motion [4] yields mathematical models of the system without any control force from hydraulic (Q^c) (as shown in Eq. (2)):

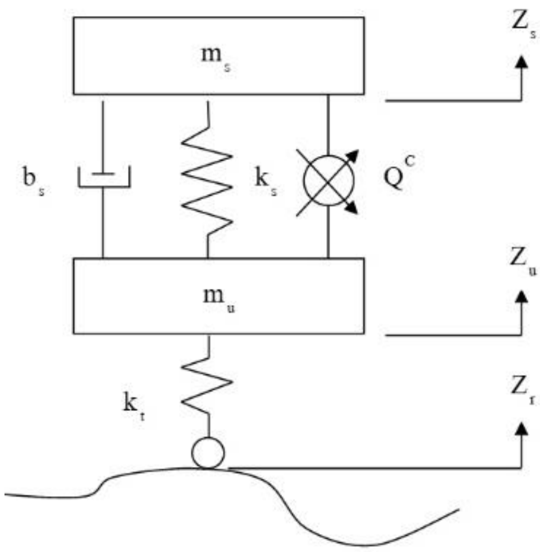

Fig.1 Quarter car

For sprung mass

$$m_s\ddot{z}_s = -k_s(z_s - z_u) - b_s(\dot{z}_s - \dot{z}_u), \quad (9)$$

and for unsprung mass

$$m_u\ddot{z}_u = k_s(z_s - z_u) + b_s(\dot{z}_s - \dot{z}_u)$$
$$k_t(z_r - z_u), \quad (10)$$

where
m_s is sprung mass (kg)
m_u is unsprung mass (kg)
z_s is displacement of sprung mass (m)
z_u is displacement of unsprung mass (m)
$\dot{z}_s$ is velocity of sprung mass (m/s)
$\dot{z}_u$ is velocity of unsprung mass (m/s)
k_s is spring constant of sprung mass (N/m)

b_s is damping constant of sprung mass(N/m^2)
k_t is spring constant of tire (N/m)
z_r is displacement from road (m).

Rearranging Eq. (9) and (10) into a matrix form, the following was obtained:

$$\begin{bmatrix} m_s & 0 \\ 0 & {}_u \end{bmatrix}\begin{bmatrix} \ddot{z}_s \\ \ddot{z}_u \end{bmatrix} = \begin{bmatrix} k_s(z_u - z_s) + b_s(\dot{z}_u - \dot{z}_s) \\ k_s(z_s - z_u) + b_s(\dot{z}_s - \dot{z}_u) + k_t(z_r - z_u) \end{bmatrix}. \tag{11}$$

The matrix in front of the acceleration is considered as the mass matrix (M) and another one on the right side of the equation is the generalized force vector (Q). Therefore, these two parameters of the unconstrained system are obtained:

$$M(q,t) := \begin{bmatrix} m_s & 0 \\ 0 & m_u \end{bmatrix},$$

$$Q(q,\dot{q},t) := \begin{bmatrix} k_s(z_u - z_s) + b_s(\dot{z}_u - \dot{z}_s) \\ k_s(z_s - z_u) + b_s(\dot{z}_s - \dot{z}_u) + k_t(z_r - z_u) \end{bmatrix}. \tag{12}$$

3.2 Assigning a Constraint Equation

This step is to define a function that controls the system. Let us begin with the constraint equation from Eq. (4). Since the movement stability of the sprung and unsprung masses is considered, Lyapunov stability criterion is used in this study [7].

Regarding Lyapunov stability criterion, it composes of a function of V , which is also known as a Lyapunov candidate function. In the present work, it is assumed as a function of displacement, velocity and time. Stability in Lyapunov sense is satisfied with two conditions:

1. *A positive definite function of the Lyapunov candidate*

$$V(q,\dot{q},t) > 0. \tag{13}$$

2. *A negative definite function of the Differentiation of the Lyapunov candidate*

$$\frac{d}{dt}(V(q,\dot{q},t)) < 0. \tag{14}$$

Satisfying the first condition is not a challenge; however, it is not easy to find a positive definite function having negative definite differential function. Thus, any positive definite function V that satisfies

$$\dot{V}(q,\dot{q},\ddot{q},t) = -\alpha V(q,\dot{q},t) \tag{15}$$

as a constraint is considered.

In Eq. (15), V is a Lyapunov candidate which is a function of generalized coordinate, generalized velocity and time. α is any positive constant parameter which is to show the speed where the system can return to equilibrium.

The purpose of this study is to reduce the system's energies and return the displacement and velocity of the sprung and unsprung masses to equilibrium. Eq. (15) is solved to prove that this constraint equation satisfies the goal. So, we can get the solution of this equation as:

$$V = e^{-\alpha t} V_0. \tag{16}$$

This equation is in the form of an exponential decay. When the time goes to · , the function will bring V back to zero. So, the parameters (q and $\dot{q}$) that are defined as the function of V are also brought back to zero, which is the equilibrium point.

In the present study, the Lyapunov candidate function [7] is used as:

$$V = \frac{}{2} a_1 q^T q + \frac{1}{} a_2 v^T v + a_{12} q^T v, \tag{17}$$

where a_1 , a_2 , a_{12} are positive constants that satisfy

$a_1 > 0,\ a_2 > 0$ and

$$0 < a_{12} < \sqrt{a_1 a_2}, a = \frac{2a_{12}}{a_2}$$

(as shown in Eq. (15)),

$$q := \begin{bmatrix} z_s \\ z_u \end{bmatrix},\ v := \dot{q} := \begin{bmatrix} \dot{z}_s \\ \dot{z}_u \end{bmatrix},$$

$$a := \ddot{q} := \begin{bmatrix} \ddot{z}_s \\ \ddot{z}_u \end{bmatrix},$$

where q is the generalized displacement, v is the generalized velocity, and a is the generalized acceleration. The derivatives of Eq. (17) are:

$$\dot{V} = a_1 \begin{bmatrix} z_s & z_u \end{bmatrix} \begin{bmatrix} \dot{z}_s \\ \dot{z}_u \end{bmatrix} + a_2 \begin{bmatrix} \dot{z}_s & \dot{z}_u \end{bmatrix} \begin{bmatrix} \ddot{z}_s \\ \ddot{z}_u \end{bmatrix} + a_{12} \begin{bmatrix} \dot{z}_s & \dot{z}_u \end{bmatrix} \begin{bmatrix} \dot{z}_s \\ \dot{z}_u \end{bmatrix} + a_{12} \begin{bmatrix} z_s & z_u \end{bmatrix} \begin{bmatrix} \ddot{z}_s \\ \ddot{z}_u \end{bmatrix}. \quad (18)$$

From this part, Eq. (17) and Eq. (18) are placed into Eq. (15):

$$a_1 \begin{bmatrix} z_s & z_u \end{bmatrix} \begin{bmatrix} \dot{z}_s \\ \dot{z}_u \end{bmatrix} + a_2 \begin{bmatrix} \dot{z}_s & \dot{z}_u \end{bmatrix} \begin{bmatrix} \ddot{z}_s \\ \ddot{z}_u \end{bmatrix} + a_{12} \begin{bmatrix} \dot{z}_s & \dot{z}_u \end{bmatrix} \begin{bmatrix} \dot{z}_s \\ \dot{z}_u \end{bmatrix} + a_{12} \begin{bmatrix} z_s & z_u \end{bmatrix} \begin{bmatrix} \ddot{z}_s \\ \ddot{z}_u \end{bmatrix} = -\alpha V. \quad (19)$$

Rearranging Eq. (19) into the acceleration form (as shown in Eq. (5)), we have:

$$\begin{bmatrix} a_2 \dot{z}_s + a_{12} z_s & 0 \\ 0 & a_2 \dot{z}_u + a_{12} z_u \end{bmatrix} \begin{bmatrix} \ddot{z}_s \\ \ddot{z}_u \end{bmatrix} = -\begin{pmatrix} a_1 z_s \dot{z}_s + a_1 z_u \dot{z}_u \\ + a_{12} \dot{z}_s^{\,2} + a_{12} \dot{z}_u^{\,2} \end{pmatrix} - \alpha V. \quad (20)$$

From Eq. (20), parameters A and b which are the constraint matrix and the constraint vector, respectively, are found as:

$$A := \begin{bmatrix} a_2 \dot{z}_s + a_{12} z_s & 0 \\ 0 & a_2 \dot{z}_u + a_{12} z_u \end{bmatrix},$$

$$b := -\begin{pmatrix} a_1 z_s \dot{z}_s + a_1 z_u \dot{z}_u \\ + a_{12} \dot{z}_s^{\,2} + a_{12} \dot{z}_u^{\,2} \end{pmatrix} - \alpha V. \quad (21)$$

3.3 Computing Control Forces of the Constrained System

Substituting M, Q from the unconstrained system and A, b from the constraint equation with Eqs. (6) and (7), the dynamic equation of the constrained suspension system with control forces was:

$$M\ddot{q} = Q(q, \dot{q}, t) + A^T (AM^{-1}A^T)^+ (b - Aa) \quad (22)$$

Having the dynamic equation of the constrained system, its displacement and velocity can be verified in any period of time by using MATLAB to compute and validate the results.

4. RESULT

The parameters of the system and the controller are considered as m_s = 208 kg, m_u = 28 kg, k_s = 18709 N/m, k_t = 127200 N/m, b_s = 1300 Ns/m, a_1 = 1, a_2 = 8, a_{12} = 1, and α = 0.25.

Given the system's mathematical model, both masses are considered to have separate equilibrium points related to CG of their particular masses.

In the experiment, the values of a_1, a_2, a_{12}, and α are varied until the suitable values are found.

In the simulation, the values of the initial conditions of the system are defined as [z_s = 0, z_u =0, $\dot{z}_s$ = 0.01, $\dot{z}_u$ = 0.05].

These values might not be the correct initial values related to the constraint equation of the system. Therefore, the nonlinear solver *fsolve* command in MATLAB is used to solve the right initial conditions of the non-linear system. The initial conditions that met with the constraint equation were found as z_s = -0.0028, z_u = -0.0140, $\dot{z}_s$ = 0.0014, $\dot{z}_u$ = 0.0070.

It could be seen that the initial conditions have changed since it is calculated with the equation related to the constraint of the system. Therefore, the accurate initial conditions are received and could be used in the next step of calculation.

In this work, the proposed active suspension controller is tested with three types of road:

1. A sinusoidal wave road with the height of slope equals to 0.075 meters and frequency equals to 0.242 Hz (as shown in Fig.2). This model represents a normal road condition.

2. A sinusoidal wave road with natural frequency with the height of slope equals to 0.015 meters and frequency equals to 0.625 Hz, the value of natural frequency of the system (as shown in Fig.5). This case shows that the model has the least suitable frequency for the car of the designed system.

3. A random road with all random heights of the slope and frequency by MATLAB function *rand* (as shown in Fig.8) yields the example model of an unexpected road type.

The results only show the displacements of the sprung mass of each road type (as shown in Figs.3,6, and 9), which are comparable to the trembling in the car cabin and the additional control forces (as shown in Figs.4, 7, and 10) and they negate with the given forces of the system.

Figs.3, 6, and 9 show that the sprung mass which is considered as a car cabin goes back to the equilibrium set point within a short period of time while Figs.4,7, and 10 verify the possibility in generating control forces.

If the force which controls the system is close to the force generated by the system, there is a possibility that we could in reality control the system using the calculated control force.

4.1 Response from Sinusoidal Wave

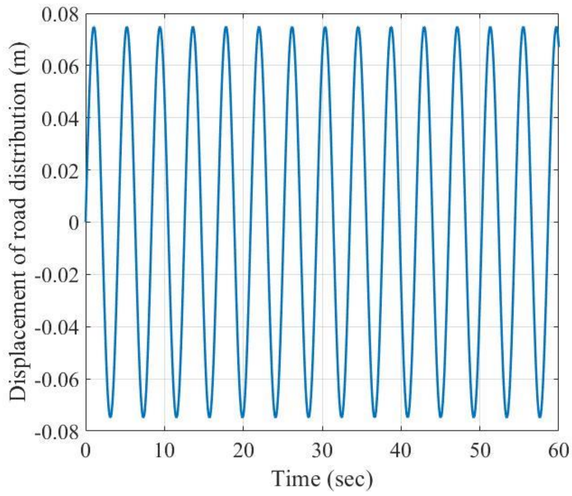

Fig.2 Road distribution of sinusoidal wave input

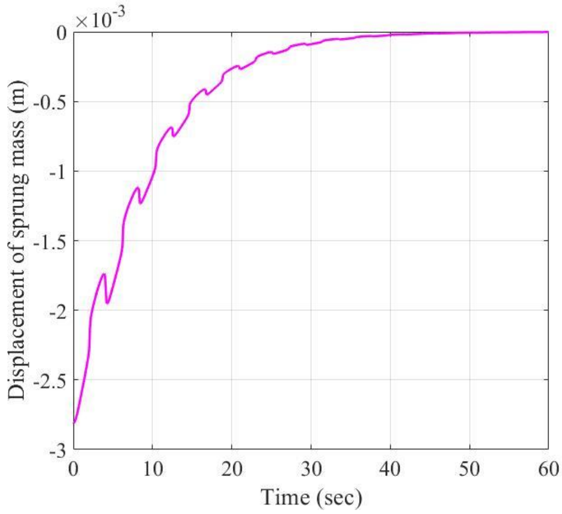

Fig.3 Displacement of sprung mass of sinusoidal wave input

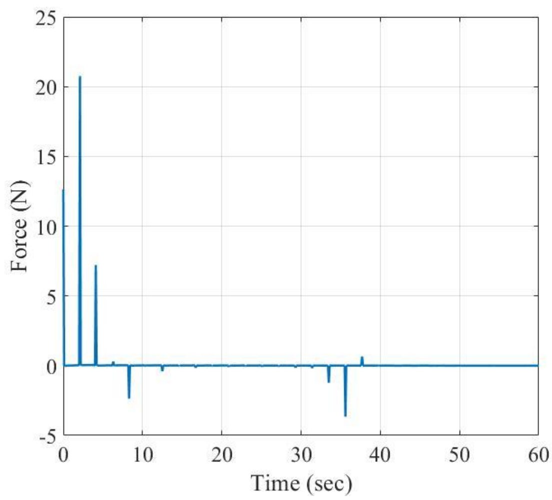

Fig.4 $^{C} - Q$ of sinusoidal wave input

4.2 Response from Sinusoidal Wave with Natural Frequency

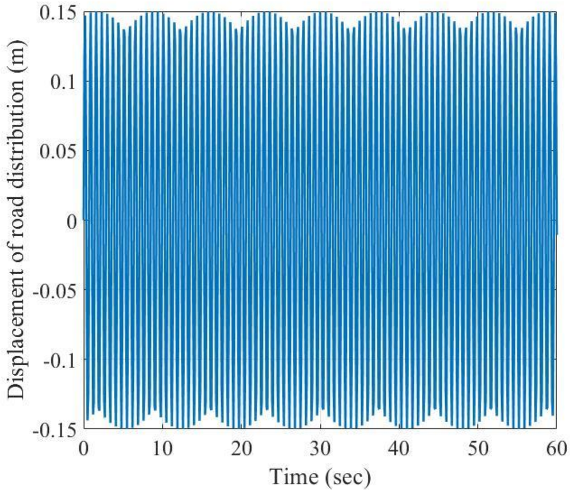

Fig.5 Road distribution of sinusoidal wave with natural frequency input

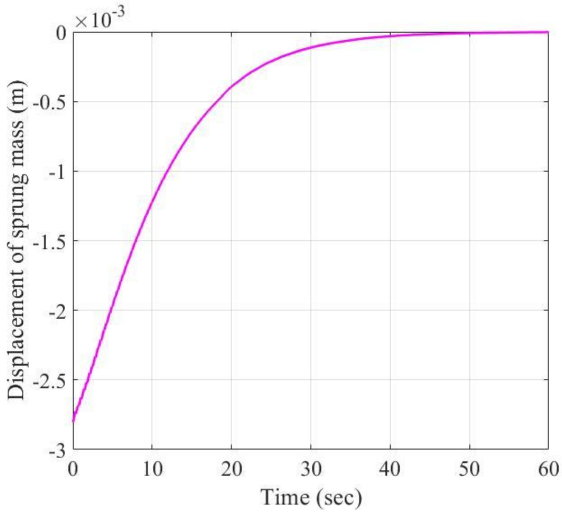

Fig.6 Displacement of sprung mass of sinusoidal wave with natural frequency input

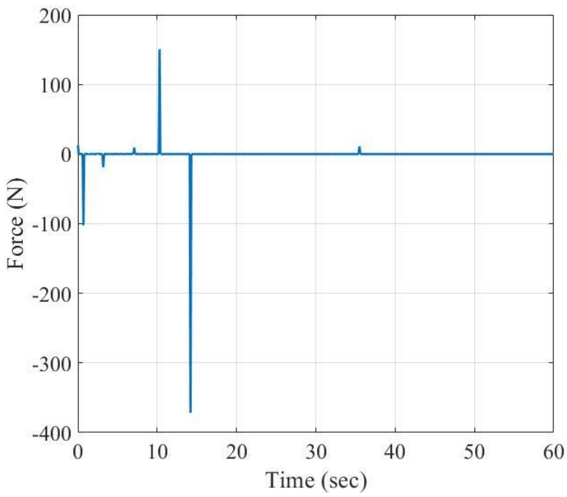

Fig.7 $Q^C - Q$ of sinusoidal wave with natural frequency input

4.3 Response from Random Road Distribution

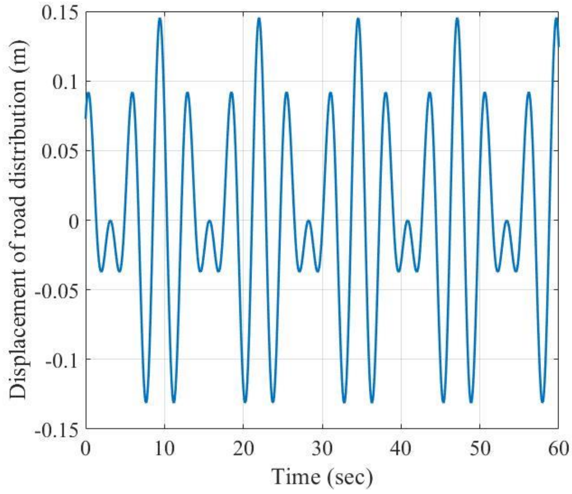

Fig.8 Road distribution of random road distribution input

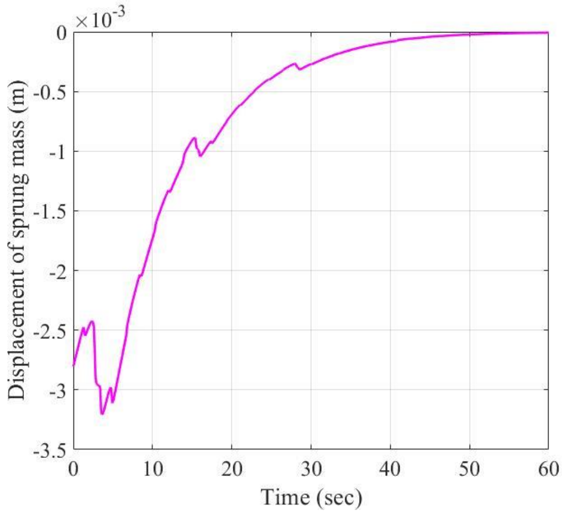

Fig.9 Displacement of sprung mass of random road distribution input

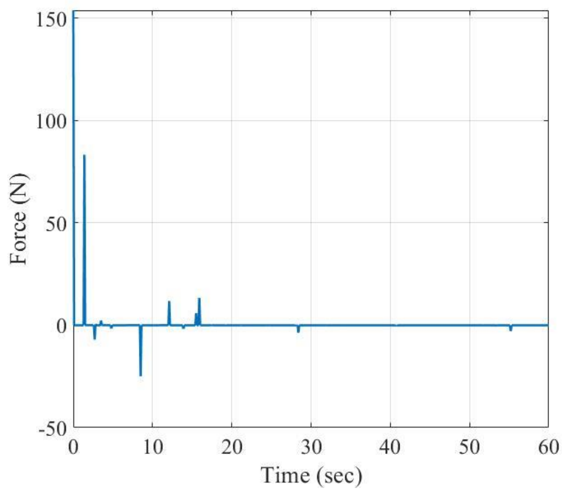

Fig.10 $Q^C - Q$ of random road distribution input

In addition, to conclude our assumption on choosing α , we try to increase the value of α to 1.5 (the original was 0.25).

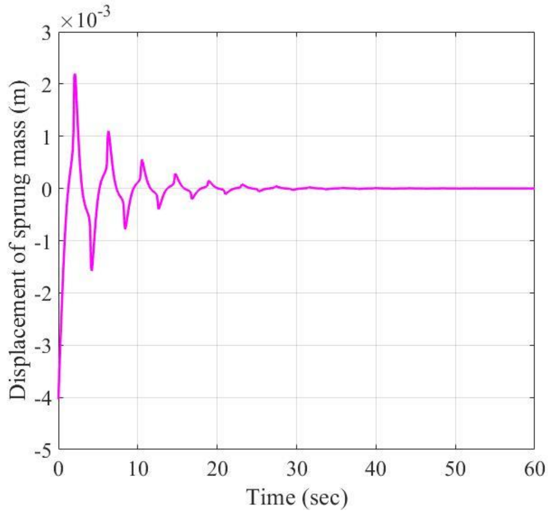

Fig.11 Displacement of sprung mass of sinnuisodal wave input with $\alpha=1.5$

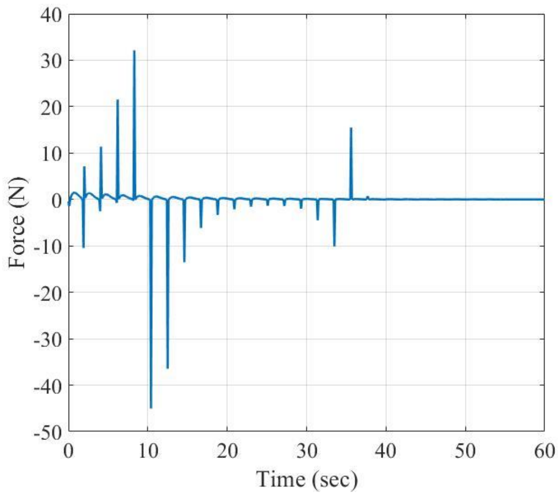

Fig.12 $^{C}-Q$ of sinnuisodal wave input with $\alpha=1.5$

The results show that the value of $Q^C - Q$ is increased for the same sinusoidal wave input (Observing from the comparison between Fig.12 and Fig.4) and the displacement of the sprung mass returns to its equilibrium faster (Noticing from the comparison between Fig.11 and Fig.3). However, this result trades-off with the bigger control force Q^C .

5. CONCLUSION

In this work, the fundamental equations of constrained motion are applied to a quarter active suspension car. The Lyapunov Stability Theory is used in the constraint equation because this theory is based on energy reduction of the system to its equilibrium. Eventually, the dynamic equation of the constrained system that guarantees constraint-following is achieved. Then, the displacement results of the sprung and unsprung masses are verified by numerically computing the dynamic constrained equation using MATLAB Program.

The results showed that the displacements of the sprung mass and unsprung mass could return to the equilibrium. Moreover, with the experimentation, it is found that the speed of the system when it reaches the equilibrium point could be adjusted by varying the constant parameter α . The greater of α makes the system return to equilibrium faster but the control forces (Q^C) might increase respectively to α . The bigger the control force is from the generalized force, the less opportunity is there to succeed in applying the physical part to the suspension system. Therefore, the best solution is to choose a suitable value of α , which complies with physical parts while still maintaining the efficiency of the controller.

Future research can be conducted on simulations with many other road types and more realistic inputs. Also, equipment that can work in accordance with the designed controller can be developed to make this suspension system applicable to real life.

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