ON THE CONSTRUCTION OF THE NODAL EXACT FINITE ELEMENT MODEL FOR AXIALLY LOADED PILE IN ELASTO-PLASTIC SOIL

* Chinnapat Buachart and Chayanon Hansapinyo

Faculty of Engineering, Graduate School, Chiang Mai University, Thailand

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ABSTRACT: An iterative algorithm to construct a displacement based finite element method for analyzing axially loaded pile embedded in finite depth of elasto-plastic soil is presented. The investigation herein is conducted on the condition of shape function by which exact solutions may be reproduced at the nodal points regarding to a few number of elements. The examined shape functions which satisfy the homogeneous governing equations in elastic and plastic soil through bisection iterative algorithm are introduced to obtain the so-called exact element stiffness matrix via total potential energy principle. Numerical examples of elastostatic pile embedded in elasto-plastic Winkler foundation illustrate the accuracy of proposed element compare with conventional finite element shape functions. Axial force and displacement solutions show very good agreement with data from the available literature i.e. the exact nodal displacement solution is obtained correspond to any point load level even with a single element mesh employed.

Keywords: Axially Loaded Pile, Displacement Method, Finite Element, Iterative Algorithm, Soil-Pile Interaction

1. INTRODUCTION

In this study, the displacement of axially loaded pile embedded in elasto-plastic soil is solved via proposed finite element procedure. The nodal exact shape function concept suggested in [1]–[4] are used to construct the tangent stiffness matrix and equivalent nodal force incorporate with Newton-Rapshon algorithm to solve nonlinear algebraic equations. An internal point which is separate elastic and plastic portion soil around embedded pile then determined via simple iterative bisection procedure.

Example of elasto-static pile embedded in elasto-plastic soil subjected to quasi-static point load on top soil level is analyzed. The results from proposed element are compared with analytical solution from [5] and conventional simplex bar element to verify the accuracy of proposed pile element.

2. MATHEMATICAL FORMULATION

2.1 Problem Definition

The analysis considers a single circular pile [4], with diameter d (cross section $A = 0.25\pi d^2$ and perimeter $U = \pi d$), embedded into soil deposit (Fig. 1). The pile has a total length L and is subjected to an axial force P_0 at the pile head which is flush with the ground surface. The soil medium is assumed to be elastio-plastic, isotropic and homogeneous, with elastic properties described by equivalent spring coefficient k_s. Once the soil displacement go beyond the yielding displacement w_*, the shear resistance will keeps constant as long as the displacement increases. The soil bearing capacity at pile's end is presented by coefficient k_b. The pile is assumed to behave as an elastic column with Young's modulus E. The Poisson's ratio of the pile material is neglected. Figure 1(a) shows the elastic and plastic zone occurred in soil due to the axial displacement w_0 at top pile head is greater than yielding displacement w_*. Suppose that yielding displacement occurred at depth z_0 from pile head. Hence, the length of an elastic portion in Fig. 1 is denoted by ℓ, the magnitude of axial displacement in elastic portion is less than or equal to yielding displacement.

2.2 Governing Differential Equations

Consider one-dimensional element in Fig. 2. Assuming that soil surrounding pile element is in elastic condition for whole length. The total potential energy of this soil-pile element subjected to the axial forces P_1 and P_2 is defined as the sum of internal potential energy (strain energy) and the external potential energy due to external load as follow [4]:

$$\Pi = \frac{1}{2}\left[EA\int_0^L \left(\frac{dw}{dz}\right)^2 dz + k_s U \int_0^L w^2 dz \right] - P_1 w_1 - P_2 w_2 \quad (1)$$

where $w(z)$ is the vertical pile displacement at depth z. The first and second terms in Eq. (1) represent the strain energy in pile and surrounding soil, respectively.
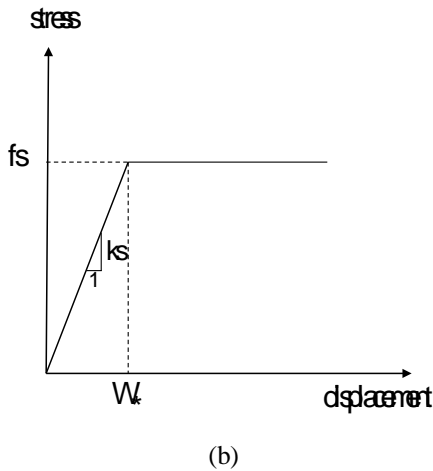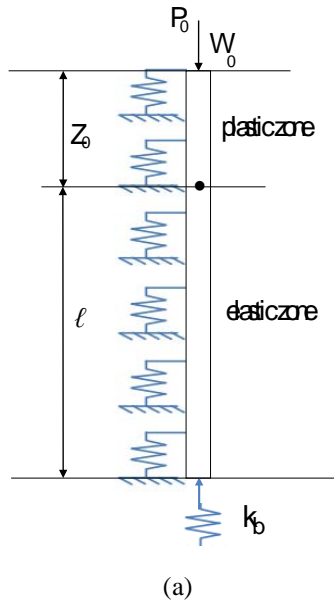

(a)

(b)

Fig. 1   Axially loaded pile and soil model [4]

The first variation of Eq. (1) leads to

$$\delta\Pi = EA\int_0^L \left(\frac{d\delta w}{dz}\right)\left(\frac{dw}{dz}\right)dz + k_s U \int_0^L (\delta w)w\,dz \qquad (2)$$
$$- P_1\delta w_1 - P_2\delta w_2$$

Applying the appropriate Gauss-Green theorem to Eq. (2) and setting $\delta\Pi = 0$, gives the differential equation for equilibrium

$$\frac{d^2w}{dz^2} - \alpha^2 w = 0 \quad \text{for} \ 0 < z < L \qquad (3)$$

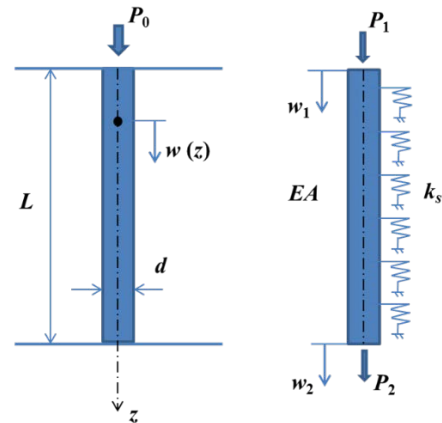where $\alpha^2 = k_s U / EA$ is the characteristic parameter of pile surrounding by elastic soil region.

Fig. 2   Axially loaded pile element [4]

A set of natural boundary conditions are as follows

$$P_1 = -EA\frac{dw}{dz}\bigg|_{z=0} \quad \text{and} \ \ P_2 = EA\frac{dw}{dz}\bigg|_{z=L} \qquad (4)$$

Plastic soil conditions

Suppose that ended load at top soil level reach some limit, the soil portion at level $0 < z < z_0$ is in plastic condition. Consider the plastic soil portion, upper portion in Fig. 1, displacement w in second term of equilibrium equation (3) have to be replaced by yielding displacement w_* as follow

$$\frac{d^2w}{dz^2} - \frac{f_s}{EA} = 0 \ \ \text{for} \ 0 < z < z_0 \qquad (5)$$

In which $f_s = k_s U w_*$ is the maximum shearing force occurred in soil surrounding pile element. Shearing force in soil surrounding pile element is linearly dependent on pile displacement until they reach some limit of yielding displacement magnitude w_*.

The nodal exact finite element solutions of Eqs. (3) and (5) for a given value of plastic soil depth z_0 can be found in [4]. In present work, the proposed finite element procedure to solve Eqs. (3) and (5) for a given load P_0 (Fig. 1) will be explained in next section. Then, the results from proposed nodal exact element and conventional linear element will be compared with available analytical solution in [5].

2.3 Newton-Raphson Iteration

Due to elasto-plastic behavior of shear force in soil surrounding pile element, finite element discretization of Eqs. (3) and (5) leads to the steady steady-state set of non-linear algebraic equations given by the residual equation as follows [6]–[9]:

$$\mathbf{r}^{(k)} = \mathbf{f} - \mathbf{F}\left(\mathbf{w}^{(k)}\right) \qquad (6)$$

In which internal force **F** is a non-linear function (bilinear function in this case) of nodal displacement **w** at iteration number k. The external nodal load **f** is presented in term of specified vector. A solution of residual $\mathbf{r} = \mathbf{0}$ is obtained by solving an algebraic equation:

$$\mathbf{K}_T^{(k)} \Delta\mathbf{w}^{(k)} = \mathbf{r}^{(k)} \qquad (7)$$

where $\mathbf{K}_T^{(k)} = \partial\mathbf{F}/\partial\mathbf{w}^{(k)}$ is so-called tangential stiffness matrix, used to obtain the incremental nodal solution $\Delta\mathbf{w}$. Then update the nodal solution:

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} + \Delta\mathbf{w}^{(k)} \qquad (8)$$

An iteration is repeated until the magnitude of incremental nodal solution $\Delta\mathbf{w}$ and residual **r** close to zeros.

2.4 Element Internal Force

Nodal internal force **F** in Eq. (6) can be derived from first variation of strain energy term in Eq. (2). In conventional linear finite element, the trial solution $w(z)$ and variation $\delta w(z)$ are approximated via linear shape functions $N_a(z)$, $a = 1$ or 2, which results in non-exact nodal solution **w**. To produce the nodal exact solution of unknown **w**, the trial solution $w(z)$ and variation $\delta w(z)$ are approximated via the set of shape function $N_a(z)$ which satisfy homogeneous solutions of governing equations (3) and (5) as discussed in previous work [4]. Hence, nodal internal force of conventional and proposed two nodes element can be expressed as follow.

2.4.1 Plastic soil conditions

Consider the upper part of soil (plastic zone) in Fig. 1(a). The displacement of soil surrounding pile element is in plastic condition, *.i.e.* $w(z) \geq w_*$. The internal force of proposed element can be expressed as follow:

$$\mathbf{F} = \frac{EA}{z_0}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_* \end{Bmatrix} + \frac{f_s z_0}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad (9)$$

Note that the internal force Eq. (9) is derived from linear shape function as shown in appendix

2.4.2 Elastic soil conditions

Now consider the lower part of soil (elastic zone) in Fig. 1(a). The displacement of soil surrounding pile element still be in elastic condition, *.i.e.* $w(z) \leq w_*$. Hence, trial solution of pile in elastic soil condition was in hyperbolic functions as described in [4]. The internal force of proposed element can be expressed as follow:

$$\mathbf{F} = \alpha EA\begin{bmatrix} \coth\beta & -\operatorname{csch}\beta \\ -\operatorname{csch}\beta & \coth\beta \end{bmatrix}\begin{Bmatrix} w_* \\ w_2 \end{Bmatrix} \qquad (10)$$

In which non-dimensional parameter $\beta = \alpha\ell$.

2.4.3 Conventional linear element

To compare the accuracy and efficiency of proposed element, the internal force of conventional linear finite element has to be recalled. An expression of internal force vector for conventional linear finite element in elasto-plastic soil condition is shown in Eq. (11).

$$\begin{aligned}
\mathbf{F} &= \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} \\
&+ \frac{\alpha^2 EA}{6L^2}\begin{bmatrix} 2\ell^3 & \ell^2(L+2z_0) \\ \ell^2(L+2z_0) & 2(L^3 - z_0^3) \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} \\
&+ \frac{f_s}{2L}\begin{Bmatrix} z_0(L+\ell) \\ z_0^2 \end{Bmatrix}
\end{aligned} \qquad (11)$$

Then the procedure to construct tangent stiffness matrix used to determine incremental displacement following Eq. (7) will be explained.

2.5 Tangential Stiffness Matrix

Tangential stiffness matrix in Eq. (7) will be constructed from matrices of internal force in Eqs. (9) and (10). First, nodal internal forces from Eq. (9) and (10) are assembled to form three members vector, following degree of freedom w_1, w_2 and w_*. Then degree of freedom corresponding to yielding displacement w_* will be *condensed* out via static condensation technique [7]. Hence, tangential stiffness matrix is *reduced stiffness* represents a set of two equations with unknowns corresponding to ended displacements, w_1 and w_2.

2.6 Estimation of plastic zone

Construction of the internal force term in Eqs. (9)–(11) require the estimated length of plastic zone (z_0) at present iteration. Let assume that the yielding of soil surrounding pile element occurs in upper zone of pile element in Fig. 2, *i.e.* $w_1 > w_2$ and $w_2 <$

$w_* < w_1$. Imply the continuity of internal axial force via finite element equilibrium equation at $z = z_0$ (*ghost* node). Hence, relation between yielding displacement at given position z_0 and specified value of nodal solutions (w_1 and w_2) for proposed nodal exact element is:

$$w_* = \frac{w_1 \sinh \beta + w_2 \alpha z_0}{\left[1 + \frac{(\alpha z_0)^2}{2}\right] \sinh \beta + \alpha z_0 \cosh \beta} \quad (12)$$

Equation (12) is numerically solved via *bisection* method to obtain the length of plastic zone, namely z_0. Then the tangential stiffness matrix $\mathbf{K}_T$ can be obtained from procedure described in previous section. Note that if the continuity of force from natural boundary conditions, Eq. (7) were used to derive the relation between nodal displacements and length of plastic zone instead of finite element equilibrium equation. Then the result is similar with Eq. (12), except that the square term $(\alpha z_0)^2$ in denominator of Eq. (12) will be disappeared.

Besides, in the case of conventional linear finite element, the value of plastic length z_0 can be estimated directly from linear interpolation, *i.e.*

$$z_0 = \left(\frac{w_1 - w_*}{w_1 - w_2}\right) L$$
(13)

For a given value of point load at pile head, *i.e.* force P_0 in Fig. 1(a), iteration processes in Eqs. (7) and (8) are repeated until equilibrium conditions satisfied.

3. NUMERICAL EXAMPLES

In this section, numerical example is presented to illustrate the effectiveness of nodal exact finite element proposed in previous section. Results from proposed element then compare with conventional linear finite element. Accuracy of proposed element is verified using analytical solutions available in [5].

A bored pile was installed in the medium silt clay and the end bearing layer is sandstone. The pile length is 45 m, and the diameter $d = 1$ m. The elastic modulus of pile shaft $E = 2.2 \times 10^7$ kPa. From soil tests, the values of equivalent soil elastic coefficient $k_s = 12000$ kPa/m, the yielding displacement of soil $w_* = 2.6$ mm, and end bearing stiffness $k_b = 684000$ kPa/m [4]. The value $k_b A$ is added into the last diagonal member of stiffness matrix.

The numerical test was performed using one proposed nodal exact element and one conventional linear element. The magnitude of point load at pile head P_0 is gradually increased. The results from proposed and conventional elements are shown in Table 1. Finite element solutions of proposed element are identical to the exact analytical solutions in [5]. Additionally, the conventional linear element behaves stiffer than proposed element. Pile head displacements of conventional linear element are less than results from proposed element and analytical solution [5].

Table 1 Calculated plastic depth and settlement of pile head at any values of load P_0.

P_0(kN)	Present (Exact)		Linear FEM	
	z_0 (m)	w_0 (mm)	z_0 (m)	w_0 (mm)
2086	0	2.60	-	2.21
2951	9	3.91	9.64	3.24
3796	18	5.64	23.60	4.96
4593	27	7.71	33.49	7.29
5291	36	9.95	40.37	9.82
5807	45	11.98	45	11.98

For comparison, the load and settlement in Table are also plotted in Fig. 1. The load-settlement relations curve show nonlinear behavior when the load at pile head is gradually increased.
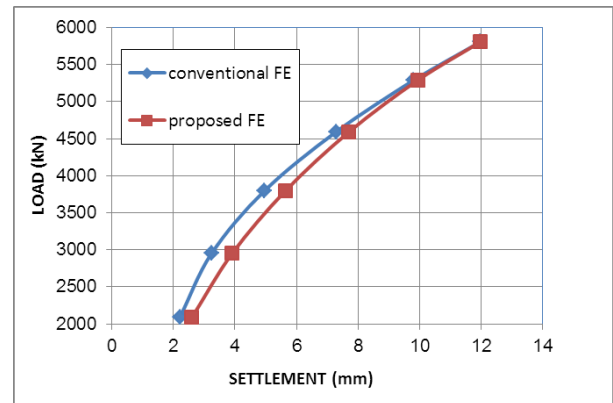

Fig. 1 Load-settlement curves of proposed versus conventional finite elements.

4. CONCLUSION

The finite element model for pile embedded in elasto-plastic soil subjected to axial load is proposed. The nonlinear algebraic equation has been solved via Newton-Raphson iteration. Tangential stiffness is constructed through static condensation technique and bisection search method. Numerical example for static load pile embedded in elasto-plastic soil was tested by gradually increase load on pile head until shear force in surrounding soil reach ultimate soil

capacity. Results obtain from proposed element were compared with analytical solution and conventional linear finite element solution. Numerical test indicates that an exact finite element solution can be obtained even with minimum number of element used. Linear conventional finite element behaved stiffer than the proposed nodal exact finite element.

5. ACKNOWLEDGEMENTS

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7. APPENDIX

7.1 Exact Solution of Pile Static Load Test

Assume that displacement at pile head in Fig. 1(a) is greater than or equals to yielding displacement w_*. Prescribe the boundary conditions (4) into the solution of governing Eqs. (3) and (5), obtain the solution of load at pile head as follows:

$$P_0 = \alpha EA w_* \left[\tanh(\alpha \ell + \gamma) + \alpha z_0 \right] \quad (14)$$

where the variable γ is the characteristic value of the end bearing stiffness of soil (k_b) at pile tip:

$$\tanh(\gamma) = \frac{k_b}{\alpha E} \quad (15)$$

The ratio of displacement at the pile head w_0 with respect to yielding displacement is therefore

$$\frac{w_0}{w_*} = 1 + \alpha z_0 \tanh(\alpha \ell + \gamma) + \frac{(\alpha z_0)^2}{2} \quad (16)$$

In addition, ratios of pile tip displacement w_t with respect to yielding displacement:

$$\frac{w_t}{w_*} = \frac{\cosh(\gamma)}{\cosh(\alpha \ell + \gamma)} \quad (17)$$

On deriving Eqs. (14)–(17), soil under pile tip is assumed to be in an elastic condition. Detail derivation of Eqs. (14)–(17) are described in reference [5].

7.2 Shape Function of Nodal Exact Element [4]

In reference [4], shape functions of proposed pile element were derived from homogeneous solutions of Eqs. (3) and (5). There are expressed in the following forms:

7.2.1 Plastic soil conditions ($0 \le z \le z_0$)

The shape functions are linear form, which is the homogeneous solution of Eq. (5):

$$N_1(z) = \frac{(z_0 - z)}{z_0}, \; N_2(z) = \frac{z}{z_0} \quad (18)$$

where z_0 is the length of plastic region in soil surrounding pile element.

7.2.2 Elastic soil conditions ($z_0 \le z \le L$)

In an elastic region, homogeneous solution of Eq. (3) is in the hyperbolic sine and cosine function. Hence, the shape functions of pile in elastic zone are in the following form:

$$N_1(z) = \frac{\sinh[\alpha(L - z)]}{\sinh \alpha \ell} \quad (19)$$

$$N_2(z) = \frac{\sinh\left[\alpha(z - z_0)\right]}{\sinh \alpha \ell} \qquad (20)$$

where $L = z_0 + l$ is total length of pile element (Fig. 1(a)). Either elastic or plastic soil portions, subscripts "1" and "2" always refer to upper and lower nodes, respectively.