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ANALYTICAL SOLUTION FOR A LONG WAVES PROPAGATION IN TWO-LAYER FLUID OVER A SUBMERGED HUMP

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ABSTRACT: The study of internal waves has become important because they are associated with the energy transfer mechanism across continental shelf edges. In addition, they also can cause strong localized departures from the surrounding ocean conditions, resulting in increased shear stresses on underwater structures and large variations in acoustic transmission properties of the ocean. Therefore, the internal waves cause problems in many areas such as offshore oil recovery, acoustic propagation in the ocean, and deep water outfall. Hence, this paper is conducted to study the two-layer long-waves propagation through submerged humps using analytical solution. Equation involved in this study are mild slope equation and the methods involve are separation of variables and a series solution. In this study the effect of the geometry of the hump when water waves propagating through the humps are also studied. Besides, the density ratio also give significance effect to the surface wave elevation. An analytical solutions obtained in this study is useful in reviewing applications and condition of the wave amplitude on submerged hump.

Keywords: Internal waves, Two-layer fluid, Mild-slope equation, Long wave, Analytic solution

1. INTRODUCTION

For almost two centuries, the progression of wave modeling research has intriguing many researchers. With concerns about energy are likely to become extinct, the future energy such as renewable wave energy are become one of the interest. According to Stewart [1] there are several types of flows in the ocean, and there is a type of flow which conserves forces and restores it with respect to density in different underwater depth called internal waves. Internal waves is formed by tidal currents propagates seamounts, continental slopes and mid-ocean pits and the waves projects tidal energy. The internal waves contain a large amount of energy, produces noises at the surface of the sea, which eventually causes cusps in the sea-surface spectrum, as discovered by Munk Cartwright [2]. The study of internal waves is pioneered by Stokes [3], Lamb [4], and Keulegan [5]. The process of wave phenomena such as refraction and diffraction of ocean surface waves can generate the energy. The famous mild-slope equation to study the refraction and diffraction on the ocean wave was introduced by Berkhoff [6]. Then, this equation is widely used by many researchers [7-10] for single layer fluid and [11-13] for two-layer fluid. Recently, Harun [13] constructed the two-layer fluid for waves propagating over a bowl-pit. By extending [14] to two-layer fluid the analytical solution towards two-layer propagating waves over a submerged hump of variable depths is discussed.

2. ANALYTIC SOLUTION

This section present the derivation of the long waves propagating in two-layer fluid over a submerged hump of variables depths by utilizing the solution given by Zhu and Harun[12] and Niu and Yu[14]. Consider, a train of plane long waves propagates in two-layer fluids with constant depth, h_1 and h_2 and the densities for upper and lower layer are denoted by ρ_1 and ρ_2 is refracted by an axi-symmetric hump-shaped located on the ocean floor as depicted in Figure 1. The geometry of the hump which located at the lower layer is as described in [14] and defined by

$$h_2(r) = \begin{cases} h_2 , & r > b \\ h_0 + \beta r^{\alpha} , & r \le b. \end{cases} \tag{1}$$

Here, b is the radius of the hump in the horizontal plane, α and β are two independent parameters which control the shape of the hump. Here, β must always satisfied the relation $\beta = (h_2 - h_0)/b^{\alpha}$ and α is considered as a rational number defined by

$$\alpha = \frac{p}{q}$$
(2)

Where p and q are integers and $q \neq 0$, $p \geq 1$, $q \geq 2$.

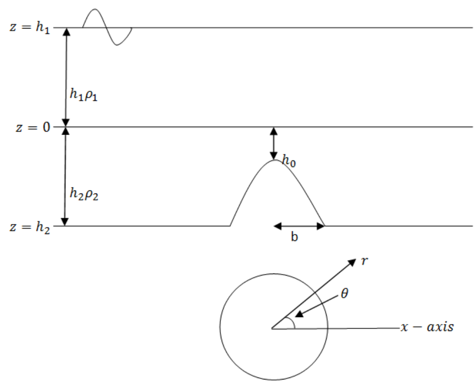

Fig. 1 definition sketch of a hump located on the floor in two-layer fluid systems.

Separation of variables

By extending the solution given by [14] to two-layer fluid, the mild-slope equation in two-layer fluid is used to find the solution. Thus, when the wavelength is much longer than the wave height, the two-layer mild slope equation can be written as [12]

$$\nabla \cdot \left(\frac{(\rho_2 - \rho_1)h_1 h_2}{\rho_1 h_2 + \rho_2 h_1} \nabla \eta\right) + \frac{\sigma^2}{g}\eta = 0$$
(3)

where

$$\sigma^2 \cong \frac{gk^2(\rho_2 - \rho_1)h_1 h_2}{\rho_1 h_2 + \rho_2 h_1}$$

and g is the gravitational acceleration, σ is the angular velocity and k is the wave number.

Because our system is axi-symmetric with respect to the z-axis, it is convenient to adopt a cylindrical coordinates system (r, θ, z) . Equation (3) can be constructed via separation of variables as

$$\eta = \sum_{n=0}^{\infty} R_n(r)\cos(n\theta)$$
(4)

with $R_n(r)$ satisfying

$$r^2(-w_1 r^{2a} - w_2 r^a - w_3)\frac{d^2 R_n}{dr^2} +$$
$$r(-w_1 r^{2a} - w_{4a} r^a - a_3)\frac{dR_n}{dr}$$
$$+((w_{5a} r^{2a} + w_{6a} r^a + w_{7a})\mu r^2 -$$
$$n^2(a_1 r^{2a} + a_2 r^a + a_3))R_n = 0$$
(5)

where

$$w_1 = (\rho_1 - \rho_2)\beta^2 \rho_1 h_1;$$
$$w_2 = (\rho_1 - \rho_2)(2h_0\rho_1 + h_1\rho_2)\beta h_1;$$
$$w_3 = h_0 h_1(\rho_1 - \rho_2)(h_0\rho_1 + h_1\rho_2);$$
$$w_{4a} = h_1(\rho_1 - \rho_2)(h_1\rho_2 a + 2h_0\rho_1 + h_1\rho_2)\beta;$$
$$w_{5a} = \beta^2 {\rho_1}^2$$
$$w_{6a} = 2\beta\rho_1(h_0\rho_1 + h_1\rho_2)$$
$$w_{7a} = (h_0\rho_1 + h_1\rho_2)^2$$
$$\mu = \frac{\sigma^2}{g}.$$

The Frobenius Method

In this paper, we only consider α as a rational number. In order to use the Frobenius method, we have to let $r = s^q$, substitute this to Eq. (5) we have

$$s^2(w_1 s^{2p} + w_2 s^p + w_3)\frac{d^2 R_n}{ds^2} +$$
$$s(w_1 s^{2p} + w_{4a} s^p + a_3)\frac{dR_n}{ds}$$
$$+((w_5 s^{2(p+p)} - w_6 s^{(p+2q)} + w_7 s^{2q}) -$$
$$q^2 n^2(a_1 s^{2p} + a_2 s^p + a_3))R_n = 0$$
(6)

where

$$w_4 = h_1(\rho_1 - \rho_2)(h_1\rho_2 p + 2h_0\rho_1 + h_1\rho_2)\beta;$$
$$w_5 = -\beta^2 {\rho_1}^2 \mu q^2$$
$$w_6 = -2\beta\rho_1(h_0\rho_1 - h_1\rho_2)\mu q^2$$
$$w_7 = -(h_0\rho_1 - h_1\rho_2)^2 \mu q^2$$

Eq. (6) is only second-order ordinary differential equation, thus, the general solution can be obtained using the Frobenius series method:

$$R_n(s) = \sum_{m=0}^{\infty} a_m s^{m+c} \tag{7}$$

where c is the root to be determined from the indicial equation and a_m are the recurrence relation to be determined and $a_0 \neq 0.$. Substituting Eq. (7) into Eq. (6) results in:

$$\begin{aligned}&\sum_{m=0}^{\infty} a_m s^{m+c}\{[(m+c)^2 - q^2n^2]w_1 s^{2p}\\&+[((m+c)(m+c-1)-q^2n^2)]s^p\\&+[(m+c)^2 - q^2n^2]w_3 + w_5\rho^{2(q+p)}\\&+w_6\rho^{p+2q} + w_7\rho^{2q}\} = 0\end{aligned} \tag{8}$$

Re-indexing Eq. (8), leads to

$$\begin{aligned}&[(m+c-2p)^2 - q^2n^2]w_1 a_{m-2p} +\\&[((m+c-p)(m+c-1)-q^2n^2)w_2\\&+(m+c-p)w_4]a_{m-p} +\\&[(m+c)^2 - q^2n^2]w_3 a_m + w_5 a_{m-2(q+p)}\\&+w_6 a_{m-(p+2q)} + w_7 a_{m-2q} = 0\end{aligned} \tag{9}$$

where $a_{m-p,} a_{m-2p}, a_{m-2(q+p)}, a_{m-(p+2q)}$ and a_{m-2q} equal to zero with the negative value of subscript. The indicial equation can be obtained by letting $m = 0$ in Eq. (9) and leads to

$$c = \pm nq \tag{10}$$

By the original variable r, these two distinct roots of the indicial equation lead to two sets of linearly independent solutions:

$$R_{1n}(r) = \sum_{m=0}^{\infty} a_m r^{m/q+n} \tag{11}$$

$$R_{2n}(r) = R_{1n}\ln(r) + \sum_{m=0}^{\infty} b_m r^{m/q-n} \tag{12}$$

Since R_{2n} becomes singular at $r = 0$, it has to be discarded with the imposition of the condition that water surface elevation must be finite at the origin.

Now, by considering Eq. (11) and when $p < 2q$, $\alpha < 2$, Eq. (9) gives

$a_m = 0$, for $0 < m < p$

$$a_m = -\frac{[((m+nq-p)(m+c-1)-q^2n^2)w_2 + (m+nq-p)w_4]a_{m-p}}{[(m+nq)^2 - q^2n^2]w_3}$$

for $p \le m < 2p$,

$$a_m = -\frac{([((m+nq-p)(m+c-1)-q^2n^2)w_2 + (m+nq-p)w_4]a_{m-p} + [(m+nq-2p)^2 - q^2n^2]w_1 a_{m-2p})}{[(m+nq)^2 - q^2n^2]w_3}$$

for $2p \le m < 2q$,

$$a_m = -\frac{([((m+nq-p)(m+nq-1)-q^2n^2)w_2 + (m+nq-p)w_4]a_{m-p} + [(m+nq-2p)^2 - q^2n^2]w_1 a_{m-2p} + w_7 a_{m-2q})}{[(m+nq)^2 - q^2n^2]w_3}$$

for $2q \le m < p+2q$,

$$a_m = -\frac{([((m+nq-p)(m+nq-1)-q^2n^2)w_2 + (m+nq-p)w_4]a_{m-p} + [(m+nq-2p)^2 - q^2n^2]w_1 a_{m-2p} + w_7 a_{m-2q} + w_6 a_{m-2q})}{[(m+nq)^2 - q^2n^2]w_3}$$

for $p+2q \le m < 2(q+p)$,

$$a_m = -\frac{\begin{array}{l}([((m+nq-p)(m+c-1)-q^2n^2)w_2 \\ +(m+nq-p)w_4]a_{m-p} \\ +[(m+nq-2p)^2-q^2n^2]w_1 a_{m-2p} \\ +w_7 a_{m-2q} + w_6 a_{m-2q}) \\ +w_5 a_{m-2(p+q)}\end{array}}{[(m+nq)^2-q^2n^2]w_3}$$

for $m \ge 2(q+p)$. (13)

Then, when $p > 2q$, $\alpha > 2$, Eq. (9) leads to

$$a_m = 0, \qquad \text{for } 0 < m < 2q,$$

$$a_m = -\frac{w_7 a_{m-2q}}{[(m+nq)^2-q^2n^2]w_3}$$

For $2q \le m < p$,

$$a_m = -\frac{\begin{array}{l}w_7 a_{m-2q} \\ +[((m+nq-p)(m+nq-1)-q^2n^2)w_2 \\ +(m+nq-p)w_4]a_{m\text{-}p}\end{array}}{[(m+nq)^2-q^2n^2]w_3}$$

for $p \le m < p+2q$,

$$a_m = -\frac{\begin{array}{l}(w_7 a_{m-2q} + \\ [((m+nq-p)(m+nq-1)-q^2n^2)w_2 \\ +(m+nq-p)w_4]a_{m\text{-}p} \\ +w_6 a_{m-(p+2q)})\end{array}}{[(m+nq)^2-q^2n^2]w_3}$$

for $p+2q \le m < 2p$,

$$a_m = -\frac{\begin{array}{l}(w_7 a_{m-2q} + \\ [((m+nq-p)(m+nq-1)-q^2n^2)w_2 \\ +(m+nq-p)w_4]a_{m\text{-}p} \\ +w_6 a_{m-(p+2q)} + \\ [(m+nq-2p)^2-q^2n^2]a_{m-2p})\end{array}}{[(m+nq)^2-q^2n^2]w_3}$$

for $2p \le m < 2(q+p)$,

$$a_m = -\frac{\begin{array}{l}(w_7 a_{m-2q} + \\ [((m+nq-p)(m+nq-1)-q^2n^2)w_2 \\ +(m+nq-p)w_4]a_{m\text{-}p} \\ +w_6 a_{m-(p+2q)} \\ +[(m+nq-2p)^2-q^2n^2]a_{m-2p} \\ +w_5 a_{m-2(p+q)})\end{array}}{[(m+nq)^2-q^2n^2]w_3}$$

for $m \ge 2(q+p)$. (14)

where m denotes the number of recurrence solutions for Frobenius series that we have to find until our solution is converged to a desire point, while n corresponds to the wave propagation modes.

Matched solution

For the general solution in the finite region with variable depth $r < b$, the water surface elevation is given by

$$\eta_2 = \sum_{n=0}^{\infty} C_n R_{1n}(k_0 r)\cos(n\theta) \qquad (16)$$

where C_n is a set of complex constant to be determined. In the constant region $r \ge b$ the solution is given by

$$\eta_1 = \eta_I + \eta_s$$
(17)

where η_I corresponds to the incident wave and η_S to the scattered wave which defined by:

$$\eta_1 = A_1 e^{ik_0 x} = A_1 \sum_{n=0}^{\infty} i^n \varepsilon_n J_n(k_0 r)\cos(n\theta)$$

(18)

$$\eta_s = \sum_{n=0}^{\infty} B_n H_{1n}(k_0 r)\cos(n\theta)$$

(19)

in which A_I is the incident wave amplitude, $i=\sqrt{-1}$, k_0 is the wave number in the constant region, J_n is the Bessel function of the first kind, B_n is a set of complex constant to be determined, H_{1n} is the Hankel function of the first kind and ε_n is the Jacobi symbol define by

$$\varepsilon_n = \begin{cases} 1, & n=0 \\ 2, & n \geq 1 \end{cases}$$

(20)

The solution in these two sub-regions must be matched on the common boundary $r=b$, require

$$\eta_1 = \eta_2 \qquad (r=b)$$

(21)

$$\frac{\partial \eta_1}{\partial r} = \frac{\partial \eta_2}{\partial r} \qquad (r=b)$$

(22)

Therefore, from Eq. ((16)-(19), the coefficients C_n and B_n can be determined as

$$B_n = \frac{-A_I i^n \varepsilon_n \begin{bmatrix} R_{1n}(b) k_0 J_n^{'}(k_0 b) \\ -R_{1n}^{'}(b) J_n(k_0 b) \end{bmatrix}}{R_{1n}(b) k_0 H_{1n}^{'}(k_0 b) - R_{1n}^{'}(b) H_{1n}(k_0 b)} \tag{23}$$

$$C_n = \frac{2A_1 i^{n+1} \varepsilon_n}{\pi b[R_{1n}(b) k_0 H'_{1n}(k_0 b) - R'_{1n}(b) H_{1n}(k_0 b)]} \tag{24}$$

in which the prime denote derivatives. By substituting these coefficients back to Eqs. (16)- (19), the water surface elevation for entire domain can be computed.

3. RESULTS AND DISCUSSION

This section presents a comparison of newly derived solution for a special case by taking $\rho_1/\rho_2 = 0$ with the single-layer equation discussed in Niu and Yu [14]. Then the effect of the density, wave height, and hump shaped to the wave refraction are examined.

Comparison with the single layer fluid

Since the 2-layer fluid model should reduce to single-layer model when $\rho_1 = 0$, it would be interesting to compare both model, as part of verification process. Let $\rho_1 = 0, \rho_2 = 5,$ $h_1 = h_2 = 3, h_0 = 2.8, b/L = 0.5$ and L =120.4. Figure 2 shows the comparison of the relative amplitudes along x-axis for two- and single-layer fluid models. As expected, both solutions are hardly distinguishable.

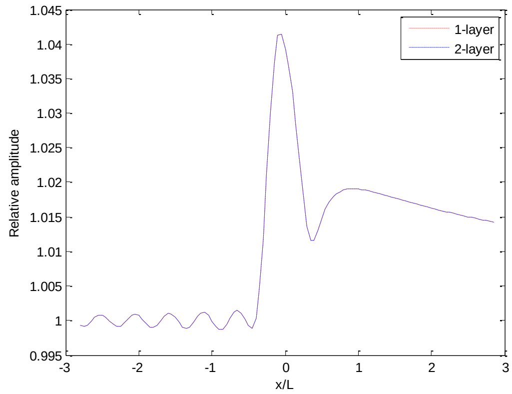

Fig. 2 Relative wave amplitude when ρ_1/ρ_2 are varied.

Effect of the density ratio

The effect of the wave refraction when the ratio of densities ρ_1/ρ_2 are varied while other parameters remain constant is discussed in this section.

Fig. 3 shows the comparisons for each value of ρ_1/ρ_2 along x -axis corresponding to ρ_1/ρ_2 =0, 1/5, 2/5, 3/5 and 4/5 while other parameters being fixed to b/L =0.5, $h_1 = h_2 = 3$. From this figure, it can be clearly seen that, the relative wave amplitude increase with the decreasing value of ρ_1/ρ_2 . This phenomenon occurs because, when the fluid is denser, the restoring force is weaker, resulting in smaller wave amplitude. Thus, we can conclude that the difference in density give very significance effect to the relative wave amplitude.

Effect of the layer thickness

Fig. 4 illustrates the relative wave amplitude for three difference values of h_1 / h_2 with ρ_1/ρ_2 =3/5. We can see that, the depth of the fluid layer also has significance effect to the relative wave amplitude. When, the upper layer is thicker, the wave can amplified more, whereas, when there is less fluid in lower layer, the relative amplitude getting smaller to balance the weight of the upper layer fluid.

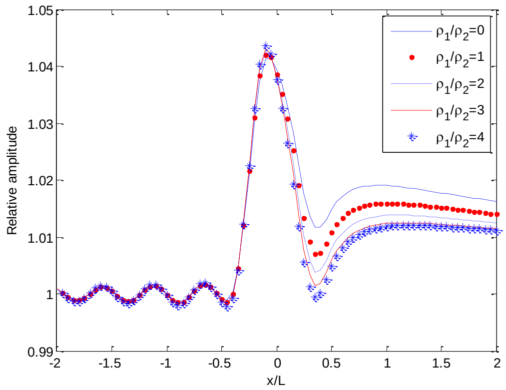

Fig. 3 Relative wave amplitude when ρ_1/ρ_2 are varied.

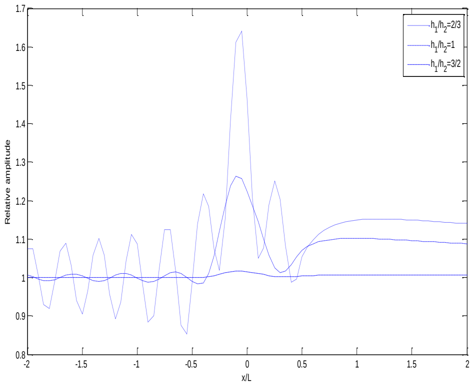

Fig. 4 Relative wave amplitude when ρ_1/ρ_2 =3/5, and h_1 / h_2 are varied.

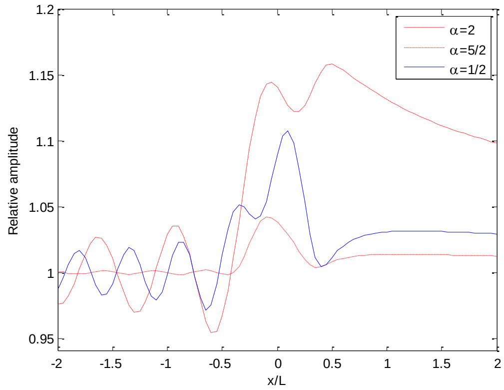

Fig. 5 Relative wave amplitude when $\rho_1/\rho_2 = 2$ and α are varied.

Effects of the hump profiles

Next, the topographic effect when the hump profiles are changed is studied. In here we set $\alpha = 1/2, 5/2$ and 2. From Fig. 5 we can observed that, the relative wave amplitude is increase with the increasing of the α that control the hump profiles. This is because, when the value of α increase, the area of the hump is expanded.

Comparison of Contour Plot between Single Two Layered Model

As mentioned earlier, this two layer model is derived and proved to be reduced into a single layer model from [14], and we have obtained the surface wave amplitude is fairly the same from each other when $\rho_1 = 0$. Now we compare the contour plots from both model with the fixed value of p and q. From the observation, there are more distortion of surface wave elevation near the hump of the two layered waves against the observation from [14] as shown in Fig. 6 with the value of density ratio $\rho_1/\rho_2 = 2/5$ and $\alpha = 1/2$.

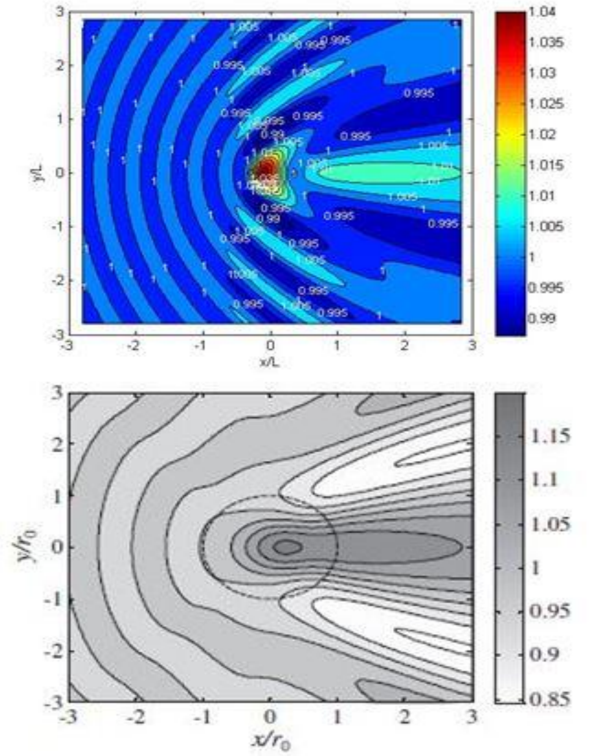

Fig. 6 *Two-Layered Model (above) in comparison to Niu and Yu (2011) results with* $\alpha = 1/2$

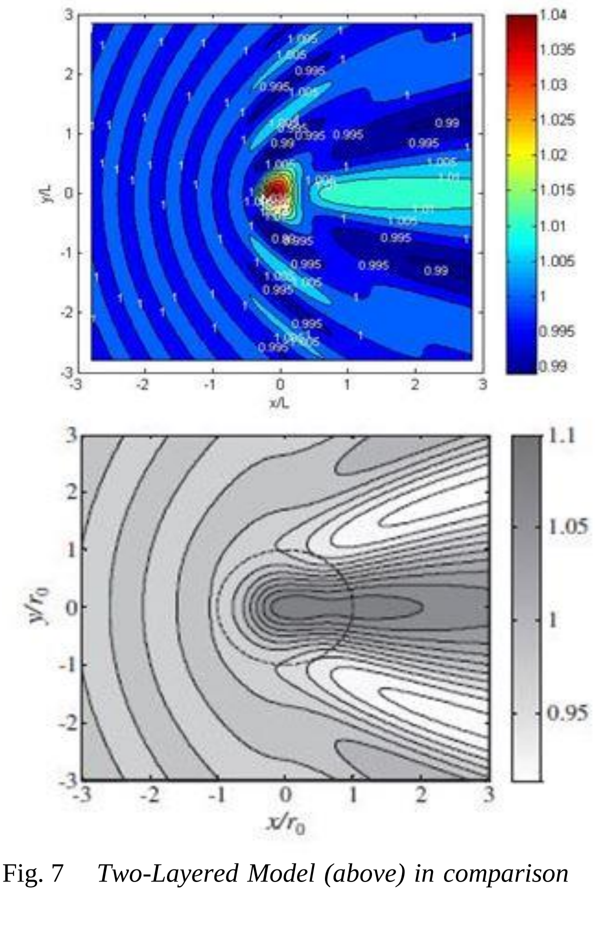

Fig. 7 *Two-Layered Model (above) in comparison to Niu and Yu (2011) results with* $\alpha = 1$

By using the same value of α with the increasing value of the density ratio $\rho_1/\rho_2 = 4/5$ as seen in Fig. 7, concludes the distortion of wave columns, both creating arrow-like shaped wave rather than smooth figures obtained in [14]. The distortion event occurs when the internal waves of the lower layer is already have its own wave amplitude transferred through energy conservation to the upper layer system and the energy from below amplifies and manifested through the surface wave amplitude as mentioned in [15].

4. CONCLUSION

A new analytic solution is derived for two-layer fluid model for long waves propagating over a circular hump. The solution is verified with the single layer fluid model when the density of the upper layer, $\rho_1 = 0$. There is also a significance effects to the relative wave amplitude with the addition of the density and the layers. The restoring force between both layers getting weaker with the smaller density difference and when there is more fluid in upper layer, the relative wave amplitude getting smaller to balance the weight of the fluid in the upper layer. For the effect of the hump profiles, the increasing of the α value lead to the increasing of relative wave amplitude. The comparison for single and two- layer model also confirm the distortion event occurs when the internal waves of the lower layer is already have its own wave amplitude transferred through energy conservation to the upper layer system and the energy from below amplifies and manifested through the surface wave amplitude as mentioned in [15]. This finding is useful in reviewing applications and condition of the wave amplitude on submerged hump, and the effect of internal waves towards surface waves behavior should be the main focus of the future in two-layer model research.

5. ACKNOWLEDGEMENT

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