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SOME PROPERTIES OF THE PRODUCT OF (P,Q) – FIBONACCI AND (P,Q) - LUCAS NUMBER

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ABSTRACT: Some mathematicians study the basic concept of the generalized Fibonacci sequence and Lucas sequence which are the (p,q) – Fibonacci sequence and the (p,q) – Lucas sequence. For example, Singh, Sisodiya and Ahmad studied the product of the k-Fibonacci and k-Lucas numbers. Moreover, Suvarnamani and Tatong showed some results of the (p, q) - Fibonacci number. They found some properties of the (p,q) – Fibonacci number and the (p,q) – Lucas number. There are a lot of open problem about them. Moreover, the example for the application of the Fibonacci number to the generalized function was showed by Djordjevicand Srivastava. In this paper, we consider the (p,q) – Fibonacci sequence and the (p,q) – Lucas sequence. We used the Binet's formulas to show that some properties of the product of the (p,q) – Fibonacci number and the (p,q) – Lucas number. We get some generalized properties on the product of the (p,q) – Fibonacci number and the (p,q) – Lucas number.

Keywords: Fibonacci sequence, Lucas sequence, (p,q) – Fibonacci number, (p,q) – Lucas number, Binet's formula

1. INTRODUCTION

Koshy [6] and Vajda [10] explained the Fibonacci number and Lucas number in their comprehensive works. The Fibonacci number F_n is the term of the sequence $\{F_n\}$ where each term is the sum of the two previous terms beginning with the initial values $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence $\{F_n\}$ is $\{0,1,1,2,3,5,8,...\}$. The Lucas number L_n is the term of the sequence $\{L_n\}$ where each term is the sum of the two previous terms beginning with the initial values $L_0 = 2$ and $L_1 = 1$. The Lucas sequence $\{L_n\}$ is $\{2,1,3,4,7,11,18,...\}$.

Falcon and Plaza [2] introduced the k-Fibonacci sequence $\{F_{k,n}\}$ which is defined as $F_{k,0} = 0$, $F_{k,1} = 1$ and $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ for $k \geq 1$ and $n \geq 1$. If $k = 1$, we get the classical Fibonacci sequence $\{0,1,1,2,3,5,8,...\}$. And we get the Pell sequence $\{0,1,2,5,12,29,70,...\}$ for $k = 2$.

Falcon [5] studied the k-Lucas sequence $\{L_{k,n}\}$ which is defined as $L_{k,0} = 2$, $L_{k,1} = k$ and $L_{k,n+1} = kL_{k,n} + L_{k,n-1}$ for $k \geq 1$ and $n \geq 1$. If $k = 1$, we get the classical Lucas sequence $\{2,1,3,4,7,11,18,...\}$. If $k = 2$, we get the Pell-Lucas sequence $\{2,2,6,14,34,82,198,...\}$.

The Binet's formulas for k - Fibonacci and k - Lucas numbers, see [1, 2, 5], are given by $F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}$ and $L_{k,n} = r_1^n + r_2^n$ where $r_1 = \frac{k + \sqrt{k^2+4}}{2}$ and $r_2 = \frac{k - \sqrt{k^2+4}}{2}$ are roots of the characteristic equation $r^2 - kr - 1 = 0$. We note that $r_1 + r_2 = k$, $r_1 r_2 = -1$ and $r_1 - r_2 = \sqrt{k^2+4}$.

In 2007, Falcon and Plaza [3] studied the k-Fibonacci sequence and the Pascal 2-triangle. Next, they considered the 3-dimensional k-Fibonacci spiral in [4]. Then Thongmoon [8,9] found some properties of the Fibonacci and Lucas numbers in 2009. In 2014, Singh, Sisodiya and Ahmad [7] studied the product of the k-Fibonacci and k-Lucas numbers.

In 2015, Suvarnamani and Tatong [11] showed some results of the (p,q)-Fibonacci number. Next Suvarnamani [12] proved some properties of the (p,q)-Lucas number in 2016. Moreover Raina and Srivastava [13] showed a class of numbers associated with the Lucas number in 1997. Moreover, the example for the application of the

Fibonacci number to the generalized function was showed by Djordjevic and Srivastava [14] in 2006. In this paper, we find some properties of the product of (p,q) Fibonacci and (p,q) Lucas numbers.

2. THEORITICAL BACKGROUND

2.1 The (p,q) - Fibonacci number

The (p,q) - Fibonacci sequence $\{F_{p,q,n}\}$ is defined as $F_{p,q,0}=0$, $F_{p,q,1}=1$ and $F_{p,q,n}=pF_{p,q,n-1}+qF_{p,q,n-2}$ for $p\geq 1$, $q\geq 1$ and $n\geq 2$. So, the (p,q) - Fibonacci number is the each term of the (p,q) - Fibonacci sequence.

2.2 The (p,q) - Lucas number

The (p,q) - Lucas sequence $\{L_{p,q,n}\}$ is defined as $L_{p,q,0}=2$, $L_{p,q,1}=p$ and $L_{p,q,n}=pL_{p,q,n-1}+qL_{p,q,n-2}$ for $p\geq 1,$ $q\geq 1$ and $n\geq 2$. So, the (p,q) - Lucas number is the each term of the (p,q) - Lucas sequence.

2.3 The Binet's formula

The Binet's formula for (p,q) - Fibonacci number $F_{p,q,n}$ is given by $F_{p,q,n}=\dfrac{r_1^n-r_2^n}{r_1-r_2}$ where $r_1=\dfrac{p+\sqrt{p^2+4q}}{2}$ and $r_2=\dfrac{p-\sqrt{p^2+4q}}{2}$ are roots of the characteristic equation $r^2-pr-q=0$. And the Binet's formula for (p,q) - Lucas number $L_{p,q,n}$ is given by $L_{p,q,n}=r_1^n+r_2^n$. We note that $r_1+r_2=p$, $r_1r_2=-q$ and $r_1-r_2=\sqrt{p^2+4q}$. We prove the Binet's formulas for (p,q) - Fibonacci number and (p,q) - Lucas number by mathematical induction.

3. THE PRODUCT OF (P,Q) - FIBONACCI NUMBER AND (P,Q) - LUCAS NUMBER

Theorem 1. Suppose that p, q, m and n be positive integers. We get

$$F_{p,q,m+n}L_{p,q,m}=F_{p,q,2m+n}+(-q)^m F_{p,q,n}\ .$$

Proof. Let p, q, m and n be positive integers. We have

$$\begin{aligned}
&F_{p,q,m+n}L_{p,q,m}\\
&=\left(\frac{r_1^{m+n}-r_2^{m+n}}{r_1-r_2}\right)\left(r_1^m+r_2^m\right)\\
&=\frac{r_1^{2m+n}-r_2^{2m+n}+r_1^{m+n}r_2^m-r_1^m r_2^{m+n}}{r_1-r_2}\\
&=\frac{r_1^{2m+n}-r_2^{2m+n}+\left(r_1r_2\right)^m\left(r_1^n-r_2^n\right)}{r_1-r_2}\\
&=\frac{r_1^{2m+n}-r_2^{2m+n}}{r_1-r_2}+\left(r_1r_2\right)^m\left(\frac{r_1^n-r_2^n}{r_1-r_2}\right)\\
&=F_{p,q,2m+n}+(-q)^m F_{p,q,n}\ .
\end{aligned}$$

□

Theorem 2. Suppose that p, q, m and n be positive integers. We get

$$F_{p,q,m}L_{p,q,m+n}=F_{p,q,2m+n}-(-q)^m F_{p,q,n}\ .$$

Proof. Let p, q, m and n be positive integers. We have

$$\begin{aligned}
&F_{p,q,m}L_{p,q,m+n}\\
&=\left(\frac{r_1^{m}-r_2^{m}}{r_1-r_2}\right)\left(r_1^{m+n}+r_2^{m+n}\right)\\
&=\frac{r_1^{2m+n}-r_2^{2m+n}-r_1^{m+n}r_2^m+r_1^m r_2^{m+n}}{r_1-r_2}\\
&=\frac{r_1^{2m+n}-r_2^{2m+n}-\left(r_1r_2\right)^m\left(r_1^n-r_2^n\right)}{r_1-r_2}\\
&=\frac{r_1^{2m+n}-r_2^{2m+n}}{r_1-r_2}-\left(r_1r_2\right)^m\left(\frac{r_1^n-r_2^n}{r_1-r_2}\right)\\
&=F_{p,q,2m+n}-(-q)^m F_{p,q,n}\ .
\end{aligned}$$

□

Theorem 3. Suppose that p, q, m and n be positive integers where $m>n$. We get

$$F_{p,q,m-n}L_{p,q,m+n}=F_{p,q,2m}-(-q)^{m-n} F_{p,q,2n}\ .$$

Proof. Let p, q, m and n be positive integers where $m>n$. We have

$$\begin{aligned}
&F_{p,q,m-n}L_{p,q,m+n}\\
&=\left(\frac{r_1^{m-n}-r_2^{m-n}}{r_1-r_2}\right)\left(r_1^{m+n}+r_2^{m+n}\right)
\end{aligned}$$

$$=\frac{r_1^{2m}-r_2^{2m}-r_1^{m+n}r_2^{m-n}+r_1^{m-n}r_2^{m+n}}{r_1-r_2}$$

$$=\frac{r_1^{2m}-r_2^{2m}-(r_1r_2)^{m-n}(r_1^{2n}-r_2^{2n})}{r_1-r_2}$$

$$=\frac{r_1^{2m}-r_2^{2m}}{r_1-r_2}-(r_1r_2)^{m-n}\left(\frac{r_1^{2n}-r_2^{2n}}{r_1-r_2}\right)$$

$$=F_{p,q,2m}-(-q)^{m-n}F_{p,q,2n}.$$

□

Theorem 4. Suppose that p, q, m and n be positive integers where $m > n$. We get

$$F_{p,q,m+n}L_{p,q,m-n}=F_{p,q,2m}+(-q)^{m-n}F_{p,q,2n}.$$

Proof. Let p, q, m and n be positive integers where $m > n$. We have

$$F_{p,q,m+n}L_{p,q,m-n}$$

$$=\left(\frac{r_1^{m+n}-r_2^{m+n}}{r_1-r_2}\right)(r_1^{m-n}+r_2^{m-n})$$

$$=\frac{r_1^{2m}-r_2^{2m}+r_1^{m+n}r_2^{m-n}-r_1^{m-n}r_2^{m+n}}{r_1-r_2}$$

$$=\frac{r_1^{2m}-r_2^{2m}+(r_1r_2)^{m-n}(r_1^{2n}-r_2^{2n})}{r_1-r_2}$$

$$=\frac{r_1^{2m}-r_2^{2m}}{r_1-r_2}+(r_1r_2)^{m-n}\left(\frac{r_1^{2n}-r_2^{2n}}{r_1-r_2}\right)$$

$$=F_{p,q,2m}+(-q)^{m-n}F_{p,q,2n}.$$

□

Theorem 5. Suppose that p, q, m and n be positive integers where $n > k$. We get

$$F_{p,q,m+n}L_{p,q,m+k}=F_{p,q,2m+n+k}+(-q)^{m+k}F_{p,q,n-k}.$$

Proof. Let p, q, m and n be positive integers where $n > k$. We have

$$F_{p,q,m+n}L_{p,q,m+k}$$

$$=\left(\frac{r_1^{m+n}-r_2^{m+n}}{r_1-r_2}\right)(r_1^{m+k}+r_2^{m+k})$$

$$=\frac{r_1^{2m+n+k}-r_2^{2m+n+k}+r_1^{m+n}r_2^{m+k}-r_1^{m+k}r_2^{m+n}}{r_1-r_2}$$

$$=\frac{r_1^{2m+n+k}-r_2^{2m+n+k}+(r_1r_2)^{m+k}(r_1^{n-k}-r_2^{n-k})}{r_1-r_2}$$

$$=\frac{r_1^{2m+n+k}-r_2^{2m+n+k}}{r_1-r_2}+(r_1r_2)^{m+k}\left(\frac{r_1^{n-k}-r_2^{n-k}}{r_1-r_2}\right)$$

$$=F_{p,q,2m+n+k}+(-q)^{m+k}F_{p,q,n-k}.$$

□

Theorem 6. Suppose that p, q, m and n be positive integers where $n < k$. We get

$$F_{p,q,m+n}L_{p,q,m+k}=F_{p,q,2m+n+k}-(-q)^{m+n}F_{p,q,k-n}.$$

Proof. Let p, q, m and n be positive integers where $n < k$. We have

$$F_{p,q,m+n}L_{p,q,m+k}$$

$$=\left(\frac{r_1^{m+n}-r_2^{m+n}}{r_1-r_2}\right)(r_1^{m+k}+r_2^{m+k})$$

$$=\frac{r_1^{2m+n+k}-r_2^{2m+n+k}+r_1^{m+n}r_2^{m+k}-r_1^{m+k}r_2^{m+n}}{r_1-r_2}$$

$$=\frac{r_1^{2m+n+k}-r_2^{2m+n+k}+(r_1r_2)^{m+n}(r_1^{k-n}-r_2^{k-n})}{r_1-r_2}$$

$$=\frac{r_1^{2m+n+k}-r_2^{2m+n+k}}{r_1-r_2}+(r_1r_2)^{m+n}\left(\frac{r_1^{k-n}-r_2^{k-n}}{r_1-r_2}\right)$$

$$=F_{p,q,2m+n+k}-(-q)^{m+n}F_{p,q,k-n}.$$

□

4. CONCLUSION

In this paper, we consider the (p,q) – Fibonacci sequence and the (p,q) – Lucas sequence. We used the Binet's formulas to show that some properties of the product of the (p,q) – Fibonacci number and the (p,q) – Lucas number. We get some generalized properties on the product of the (p,q) – Fibonacci number and the (p,q) – Lucas number.

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