Int. J. of GEOMATE, June, 2013, Vol. 4, No. 2 (Sl. No. 8), pp. 511-515
Geotec., Const. Mat. & Env., ISSN:2186-2982(P), 2186-2990(O), Japan

Lower Bound Limit Analysis of Strip Footings Resting on Cohesive Soils

S.M. Binesh[1], A. Gholampour[2]
[1,2] Shiraz University of Technology, Iran

ABSTRACT: A Novel lower bound approach is presented for the bearing capacity determination of strip footings resting on cohesive soils. The approach consists of the combination of lower bound limit theory and a mesh-free technique. In the presented method there is no need of mesh in the traditional sense and the constraints at the element discontinuities are omitted. A statically admissible stress field is constructed by the combination of mesh-free technique and the nodal integration scheme. The generated field is guaranteed to be lower bound by controlling the non-yielding condition at the entire domain. The solutions of the presented approach are compared with some well-known solutions, to demonstrate the efficiency and accuracy of the method.

Keywords: Lower bound, Mesh-free method, Strip footing

1. INTRODUCTION

Finding accurate prediction for the bearing capacity of strip footings is one of the fundamental problems in geotechnical engineering and, several researches are devoted to the subject [1]-[3]. An appealing choice to solve such problems is the application of numerical limit analysis (NLA). The use of NLA in soil mechanics problems returns back to the study of Lysmer [4] in which lower bound solutions have been found for different stability problems in the geotechnical engineering. Following Lysmer, Bottero et al [5] introduced a new finite element formulation for the limit analysis of soil structures. Sloan [6] combined the Bottero’s approach and the active set algorithm [7], and developed an efficient method for the NLA of plane problems in soil mechanics. Several other attempts have been devoted to the application of NLA in stability problems in geotechnical engineering [8]-[10].

Up to now, mesh-based methods such as finite element or boundary element method are mostly used in the NLA of structures. However, these methods suffer from some deficiencies which are mainly related to mesh definition. An alternative approach to get rid of such drawbacks is the implementation of mesh-free methods as discretization tools. A great deal of research has been devoted to the application of mesh-free methods in different fields of science [11]-[12], however, a few are dedicated to the NLA of structures [13]-[15]. To the Authors knowledge, there is no study on the application of mesh-free NLA in soil mechanics problems and hence, in present paper, a new mesh-free lower bound formulation is proposed for the bearing capacity of strip foundations resting on cohesive soils. In this regard a statically admissible stress field is approximated by a mesh-free method which uses the Shepard’s shape functions [16]. The stabilized nodal integration technique is adopted to establish a collocation method for equilibrium satisfaction throughout the problem domain. The soil beneath the foundation is assumed to be a cohesive material obeying the Tresca yield criterion. Based on the derived formulations, a computer code has been developed and the accuracy and efficiency of proposed method is investigated by solving an example at the end of the paper.

2. SHEPARD'S METHOD

The Shepard's method is used here for the construction of shape functions. Two distinct properties of Shepard's method make it appropriate to be used in the lower bound limit analysis (i) the shape functions constructed by Shepard's method have the Kronecker delta function property, which allows the simple imposition of boundary conditions. (ii) Shepard's shape functions satisfy the maximum principle. According to this principle, the interpolated values always lie between the maximum and minimum nodal values used for the interpolation process. A brief description of the method is presented in this section, and for or more details the reader is referred to [17].

Consider a function $F(P)$ which is defined over $P \subset R^2$. Any finite collection of distinct points in R^2 can be represented by $\{P_i\}_{i=1}^{N}$. The value of F at P_i is shown by F_i and, the Euclidean distance between P_i and the generic point P in R^2 is denoted by r (i.e. $r_i = \left[(x - x_i)^2 + (y - y_i)^2\right]^{1/2}$). Now the function U(P) can be written as

$$U(P) = \left[\sum_{i=1}^{N} F_i \left(\prod_{j \neq i} r_j^{\alpha}\right)\right] \Big/ \left[\sum_{i=1}^{N} \prod_{j \neq i} r_j^{\alpha}\right] \quad j = 1, 2, ..., N \tag{1}$$

where, α is a positive exponent which can affect the shape of interpolated function. Gordon and Wixom [17] suggested $\alpha > 1$ for smoothness of interpolated function. In present research α is assumed to be 3.

By the imposition of nodal values at the N nodes $\{P_i\}$, we have:

$$U(P_k) = F_k \qquad k = 1, 2, ..., N \tag{2}$$

The system of equations obtained from (2) leads to

$$U(P) = \sum_{i=1}^{N} F_i \varphi_i (P; P_1, P_2, ..., P_N) \tag{3}$$

Where $\varphi_i(P;P_1,P_2,...,P_N)$ is the shape function, and can be written as

$$\varphi_i(P;P_1,P_2,...,P_N)=\prod_{j\neq i} r_j^{\alpha}\Big/\left[\sum_{i=1}^{N}\prod_{j\neq i} r_j^{\alpha}\right] \tag{4}$$

3. LOWER BOUND LIMIT ANALYSIS

The lower bound theorem states that the collapse load obtained from any statically admissible stress field underestimates the true collapse load. A stress field is statically and plastically admissible if equilibrium and boundary conditions are fully satisfied and the yield condition is not violated anywhere.

3.1 Equilibrium satisfaction

A mesh-free collocation method is used here in conjunction with a smoothing technique to satisfy the equilibrium condition.

Assume a mesh-free scheme for a problem domain in which the interior domain and the boundaries are constructed by nodes. The general form of equilibrium equations in plane strain condition can be written as

$$\frac{\partial \sigma_{ij}}{\partial x_j}+b_i=0 \tag{5}$$

where σ_{ij} and b_i are the stress tensor component and unit body force respectively.

Equation (5) should be satisfied at all points in the problem domain, hence a Voronoi cell is constructed around each node (Fig.1) and the gradient of stress is smoothed over the cell as follows

$$\frac{\partial \tilde{\sigma}_{ij}}{\partial x_j}=\iint_{\Omega_L}\Psi\frac{\partial \sigma_{ij}}{\partial x_j}d\Omega \tag{6}$$

where $\tilde{\sigma}_{ij}$, Ψ and Ω_L are the smoothed stress, smoothing function and cell domain respectively. According to Chen's approach [18] the smoothing function can be written as

$$\Psi=\begin{cases}\dfrac{1}{A_L} & x\in A_L\\ 0 & x\notin A_L\end{cases} \tag{7}$$

where A_L is the area of Voronoi cell. Imposition of divergence theorem to the obtained equation from substituting (7) into (6), leads to

$$\frac{\partial \tilde{\sigma}_{ij}}{\partial x_j}=\frac{1}{A_L}\int_{\Gamma_L}\sigma_{ij}n_j d\Gamma \tag{8}$$

where Γ is the boundary of Voronoi cell and n_j is the normal unit vector in the direction of x_j. Equilibrium equation can be rewritten for the smoothed stress gradient by substituting (8) into (5) as

$$\frac{1}{A_L}\int_{\Gamma_L}\sigma_{ij}n_j d\Gamma+b_i=0 \tag{9}$$

By satisfaction of (9) at all pre-defined nodes, equilibrium condition for the entire problem domain can be achieved.

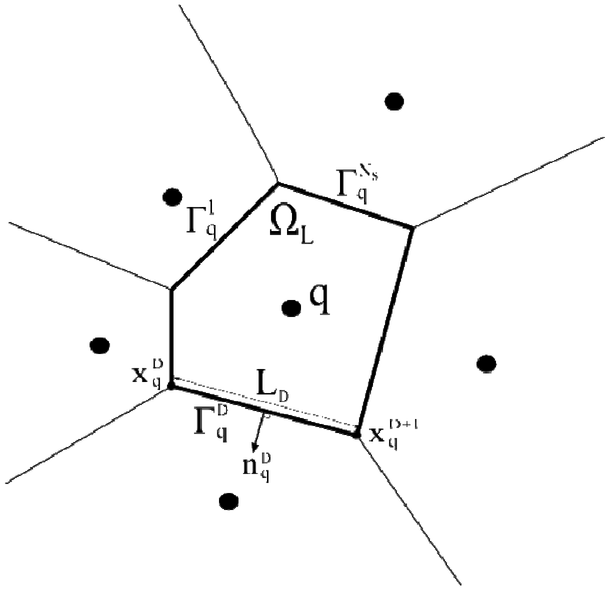

Figure 1-Voronoi cell around node q

3.2 Boundary conditions

The boundary tractions can be imposed to the problem solution by the same technique used in section 3.1 for equilibrium satisfaction. In this regard, the gradient of stress in j-direction, in each boundary node Voronoi cell, is set to be zero. According to (8) we have

$$\begin{aligned}\frac{\partial \tilde{\sigma}_n}{\partial x_j}&=\frac{1}{A_{LB}}\int_{\Gamma_B}\sigma_n n_j d\Gamma=0\\ \frac{\partial \tilde{\tau}}{\partial x_j}&=\frac{1}{A_{LB}}\int_{\Gamma_B}\tau n_j d\Gamma=0\end{aligned} \tag{10}$$

where, A_{LB} and Γ_B are related to the boundary node Voronoi cell. Knowing that the proposed shape functions have the Kronecker delta function property, the stress boundary conditions can be completely satisfied along the edge by imposing just at the boundary nodes.

3.3 Yield condition

Tresca yield criterion is adopted here for cohesive soils behavior. In plane strain condition, this criterion can be written as

$$F=(\sigma_{11}-\sigma_{22})^2-(2\sigma_{12})^2-(2S_u)^2 \tag{11}$$

where, S_u is the undrained shear strength of cohesive soil.

For a statically admissible stress field we have

$$F\le 0 \tag{12}$$

at every points in the problem domain. Equation (12) shows the locus of points located on and inside a circle in an X-Y plane where $X=\sigma_{11}-\sigma_{22}$ and $Y=2\sigma_{12}$. This circle can be approximated by a polygon of P sides (Fig.2). Thus the yield condition imposes linear inequality constrains on the stresses as follows:

$$A_K\sigma_{11}+B_K\sigma_{22}+C_K\sigma_{12}\le D \qquad K=1,2,...,P \tag{13}$$

Where

$$A_k = \cos(\frac{2\pi k}{p})$$
$$B_k = -A_k$$
$$C_k = 2\sin(\frac{2\pi k}{p}) \quad (14)$$
$$D = 2S_u \cos(\frac{\pi}{p})$$

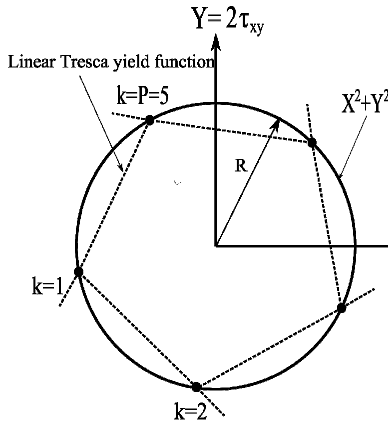

Figure 2-Linearized Tresca failure criterion

4. DISCRETE FORMULATION

According to (3), the stress values at any point $\boldsymbol{x}$ can be attributed to nodal stress values as

$$\sigma_{ij}(\boldsymbol{x}) = \sum_{z \in K} \Phi_z(\boldsymbol{x}) \sigma_{ij}(\boldsymbol{x}_z) \quad (15)$$

where, $\sigma_{ij}(\boldsymbol{x})$ is the stress value at spatial coordinate $\boldsymbol{x}$, $\Phi_z(\boldsymbol{x})$ is the shape function defined by (4), $\sigma_{ij}(\boldsymbol{x}_z)$ is the nodal stress value at the spatial coordinate $\boldsymbol{x}_z$, and K is a group of nodes located in the support domain of point $\boldsymbol{x}$.

The descritized stress field can be imposed into the required conditions for statically and plastically admissible stress field to derive the discrete form of constraints for lower bound analysis.

4.1 Equilibrium satisfaction

The relation between the smoothed stress gradient and the nodal stress values can be obtained by substituting (15) into (8) as follows:

$$\frac{\partial \tilde{\sigma}_{ij}(\boldsymbol{x})}{\partial x_j} == \sum_{z \in K} \frac{1}{A_L} \int_{\Gamma_L} \Phi_z(\boldsymbol{x}) n_j \sigma_{ij}(\boldsymbol{x}_z) d\Gamma \quad (16)$$

Substitution of (16) into (5), leads to the following matrix form

$$\boldsymbol{A}_{eq} \boldsymbol{\sigma} = \boldsymbol{B}_{eq} \quad (17)$$

where

$$\boldsymbol{A}_{eq} = [\boldsymbol{A}_1 \quad \boldsymbol{A}_2 \quad \dots \quad \boldsymbol{A}_M]^T \quad (18)$$

$$\boldsymbol{B}_{eq} = [\boldsymbol{B}_1 \quad \boldsymbol{B}_2 \quad \dots \quad \boldsymbol{B}_M]^T \quad (19)$$

$$\boldsymbol{\sigma} = [\boldsymbol{\sigma}_1 \quad \boldsymbol{\sigma}_2 \quad \dots \quad \boldsymbol{\sigma}_M]^T \quad (20)$$

where, M is the total number of nodes and, in plane strain condition, the vector of nodal stresses ($\boldsymbol{\sigma}_i$) and body forces ($\boldsymbol{B}_i$) can be written as:

$$\boldsymbol{\sigma}_i = [\sigma_{11}(x_i) \quad \sigma_{22}(x_i) \quad \sigma_{12}(x_i)]^T \quad (21)$$

$$\boldsymbol{B}_i = [b_{1i} \quad b_{2i}]^T \quad (22)$$

where, b_{1i} and b_{2i} are respectively, the unit body forces in directions 1 and 2 for node i. The configuration of matrixes $\boldsymbol{A}_1$ to $\boldsymbol{A}_M$ depends on the nodes located in the support domain of node 1 to M. Suppose that nodes r, s and t are located in the support domain of node i, then $\boldsymbol{A}_i$ can be written as

$$\boldsymbol{A}_i = [\boldsymbol{0} \;\; \dots \;\; \boldsymbol{0} \;\; \tilde{\boldsymbol{A}}_r^e \;\; \boldsymbol{0} \;\; \dots \;\; \boldsymbol{0} \;\; \tilde{\boldsymbol{A}}_s^e \;\; \boldsymbol{0} \;\; \dots \;\; \boldsymbol{0} \;\; \tilde{\boldsymbol{A}}_t^e \;\; \boldsymbol{0} \;\; \dots \;\; \boldsymbol{0}] \quad (23)$$

where

$$\tilde{\boldsymbol{A}}_{\boldsymbol{m}}^{\boldsymbol{e}} = \begin{bmatrix} A_{m1}^e & 0 & A_{m2}^e \\ 0 & A_{m2}^e & A_{m1}^e \end{bmatrix} \quad (24)$$

$$A_{m1}^e = \frac{1}{A_L} \int_{\Gamma} \Phi_m(\boldsymbol{x}) n_1 d\Gamma$$
$$A_{m2}^e = \frac{1}{A_L} \int_{\Gamma} \Phi_m(\boldsymbol{x}) n_2 d\Gamma \quad (25)$$

4.2 Boundary conditions

The descritized form of (10) can be obtained by substituting (15) into (10) as follows

$$\sum_{z \in K_B} \frac{1}{A_{LB}} \int_{\Gamma_B} \Phi_z(\boldsymbol{x}) n_j \sigma_n(\boldsymbol{x}_z) d\Gamma = 0$$
$$\sum_{z \in K_B} \frac{1}{A_{LB}} \int_{\Gamma_B} \Phi_z(\boldsymbol{x}) n_j \tau(\boldsymbol{x}_z) d\Gamma = 0 \quad (26)$$

where, K_B is a group of nodes located in the support domain of a boundary node, A_{LB} is the area of a boundary Voronoi cell and Γ_B is evaluated at the boundary Voronoi cell. Equation (26) can be written for all boundary nodes and the obtained system of equations can be assembled in the matrix form as

$$\boldsymbol{A}_{b1} \boldsymbol{\sigma} = \boldsymbol{0} \quad (27)$$

Imposition of tractions at the boundary nodes, leads to another system of equations as follows

$$\boldsymbol{A}_{b2} \boldsymbol{\sigma} = \boldsymbol{B}_{b2} \quad (28)$$

where, $\boldsymbol{A}_{b1}$ and $\boldsymbol{A}_{b2}$ are the coefficient matrixes and $\boldsymbol{B}_{b2}$ is the vector of specified values of tractions along the boundary.

The descritized form of constraints for boundary conditions can be written as

$$\boldsymbol{A}_{bo} \boldsymbol{\sigma} = \boldsymbol{B}_{bo} \quad (29)$$

where

$$\boldsymbol{A}_{bo} = \boldsymbol{A}_{b1} + \boldsymbol{A}_{b2} \;\; , \;\; \boldsymbol{B}_{bo} = \boldsymbol{B}_{b2} \quad (30)$$

4.3 Non-yielding condition

Since the discretization method (i.e. Shepard's method) Posses the maximum principle property, the non-yielding condition can be checked just at the pre-defined nodes. According to (13) the required constrains at all nodes, can be written in the matrix form as follows:

$$\boldsymbol{A}_{yi}\boldsymbol{\sigma} \le \boldsymbol{B}_{yi} \tag{31}$$

where,

$$\boldsymbol{A}_{yi} = \begin{bmatrix} \tilde{\boldsymbol{A}}_1^y & 0 & 0 & 0 \\ 0 & \tilde{\boldsymbol{A}}_2^y & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \tilde{\boldsymbol{A}}_M^y \end{bmatrix}, \qquad \boldsymbol{B}_{yi} = \begin{bmatrix} \tilde{\boldsymbol{B}}_1^y & \tilde{\boldsymbol{B}}_2^y & \dots & \tilde{\boldsymbol{B}}_M^y \end{bmatrix}^T \tag{32}$$

where

$$\tilde{\boldsymbol{A}}_k^y = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ \vdots & \vdots & \vdots \\ A_P & B_P & C_P \end{bmatrix}, \qquad \left(\tilde{\boldsymbol{B}}_k^y\right)^T = \begin{bmatrix} D & D & \dots & D \end{bmatrix}_{1\times P} \tag{33}$$

5. OBJECTIVE FUNCTION

For bearing capacity problem the objective function can be written as

$$Q = h\int_S \sigma_{nf}\, dS \tag{34}$$

where, Q is the limit load, h is the thickness normal to the plane, σ_{nf} is the normal stress acted over loaded area on the boundary and S is the length over which the normal stress is exerted. By application of the Gauss method, the integration in (34) can be written in a summation form as

$$Q = h\sum_{i=1}^{N_G} \omega_i \sigma_{nf}\left(x_G, y_G\right) \tag{35}$$

where, N_G is the number of Gauss points along S, ω_i is the weight of Gauss point i and (x_G, y_G) is the coordinate of Gauss point in (x, y) space. To determine the value of σ_{nf} in (35), a support domain is considered around each Gauss point (Fig.3). By the application of (15), the final descritized form of (35) can be written as

$$Q = \boldsymbol{C}^T\boldsymbol{\sigma} \tag{36}$$

where $\boldsymbol{C}$ is the coefficient vector and $\boldsymbol{\sigma}$ is the nodal stress vector.

6. LINEAR PROGRAMMING PROBLEM

By assembling the obtained relations for the constraints and the objective function, the problem of finding a statically and plastically admissible stress field can be written as

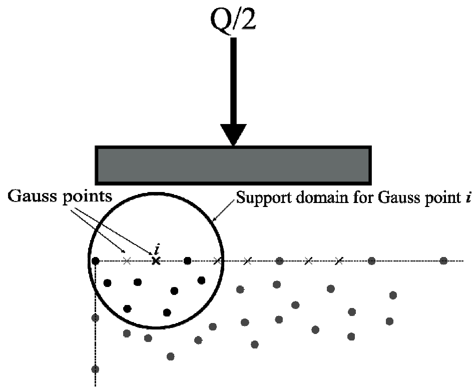

Figure 3- Support domain of a Gauss point under the loading area

$$\begin{aligned} &Minimize \quad -\boldsymbol{C}^T\boldsymbol{\sigma} \\ &Subjected\ to:\ \boldsymbol{A}_{tot}\boldsymbol{\sigma} = \boldsymbol{B}_{tot} \\ &\qquad\qquad\quad \boldsymbol{A}_{yi}\boldsymbol{\sigma} \le \boldsymbol{B}_{yi} \end{aligned} \tag{37}$$

where

$$\boldsymbol{A}_{tot} = \boldsymbol{A}_{eq} + \boldsymbol{A}_{bo}, \qquad \boldsymbol{B}_{tot} = \boldsymbol{B}_{eq} + \boldsymbol{B}_{bo} \tag{38}$$

An inbuilt library program, LINPROG, which is available in MATLAB, is used for solving above problem.

7. NUMERICAL STUDY

In this section, a smooth rigid footing, which is resting on a cohesive soil, is considered. The undrained shear strength of the soil is assumed to increase linearly with depth. The exact solution of this problem is presented by Davis and Booker [19] as follows:

$$q_f = F[(2+p)S_{u0} + rB/4] \tag{39}$$

where q_f is the bearing capacity, S_{u0} is the undrained shear strength at the ground surface, B is the footing width and ρ is the coefficient of variation of undrained shear strength with depth. F is a non-dimensional factor which depends on the footing roughness and $\frac{\rho B}{S_{u0}}$. By assuming $\frac{\rho B}{S_{u0}} = 3$ and $S_{u0} = 1$, for smooth foundation, $F = 1.22$ and hence, the exact solution for bearing capacity is 7.1858.

To solve the problem by the proposed method, a mesh-free model shown in Fig.4 is considered. The Voronoi diagram and the boundary conditions are also shown in the figure. The model has 811 nodes which are oriented in fan pattern. The support domain around each node is defined by an adjustable method to construct the shape functions. To guarantee that sufficient and suitable nodes are covered by the support domains, an automatically self-tuned value is devised in the code to adjust the radiuses of supports. The value of bearing capacity obtained from proposed method is 7.0324 which is about 2% lower than the exact solution result.

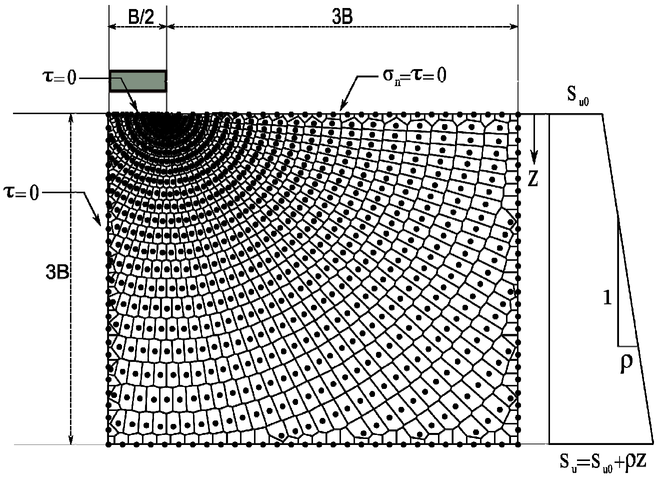

Figure 4- Problem geometry and boundary condition for 811 nodes and their Voronoi diagrams

8. CONCLUSION

A novel lower bound limit analysis formulation has been proposed for the bearing capacity determination of strip footings resting on cohesive soils. In the presented approach, there is no need of mesh in the traditional sense and, the shortcomings related to mesh definition are vanished. The results of numerical study showed very good lower bound predictions for the bearing capacity of strip footing resting on cohesive soil.

9. REFERENCES

[1] Bolton MD, Lau CK, "Vertical bearing capacity factors for circular and strip footings on Mohr–Coulomb soil," Canadian Geotechnical Journal 1993;30(6):1024–33.

[2] Frydman S, Burd HJ, "Numerical studies of bearing-capacity factor N_γ," Journal of Geotechnical and Geoenvironmental Engineering (ASCE) 1997;123(1):20–29.

[3] Kumar J, " N_γ for rough strip footing using the method of characterisitics," Canadian Geotechnical Journal 2003;40(3):669–74.

[4] Lysmer J, "Limit analysis of plane problem in soil mechanics," Soil Mechanics and Foundation Division (ASCE) 1970;96(4):1311-33

[5] Bottero A, Negre R, Pastor J, Turgeman S, "Finite element method and limit analysis theory for soil mechanics problems," Computer Methods in Applied Mechanics and Engineering 1980;22:131-49.

[6] Sloan SW, "Lower bound limit analysis using finite elements and linear programming," International Journal for Numerical and Analytical Methods in Geomechanics 1988;12:61–77.

[7] Sloan SW, "A steepest edge active set algorithm for solving sparse linear programming problems.,"International Journal for Numerical Methods in Engineering 1988;26:2671–2685.

[8] Merifield RS, Sloan W, Yu HS, "Rigorous plasticity solutions for the bearing capacity of two-layered clays," Geotechnique 1999;49(4):471-90.

[9] Kim J, Salgado R, Yu HS, "Limit analysis of soil slopes subjected to pore-water pressures," Journal of Geotechnical and Geoenvironmental Engineering, ASCE 1999;125(1):48-58.

[10] Durand AF, Vargas EA, Vaz LE, "Applications of numerical limit analysis (NLA) to stability problems of rock and soil masses," International Journal of Rock Mechanics & Mining Sciences 2006;43:408-25.

[11] Atluri SN, Zhu T, "A new meshless local Petrov-Galerkin (MPLG) approach in computational mechanics," Computational Mechanics 1998;22:117-27.

[12] Liu GR, Gu YT, "A point interpolation method for two-dimensional solids," International Journal for Numerical Methods in Engineering 2001;50:937-51.

[13] Chen S, Liu Y, Cen Zh, "Lower-bound limit analysis by using the EFG method and non-linear programming," International Journal for Numerical Methods in Engineering 2008;74:391-415.

[14] Le CV, Gilbert M, Askes H, "Limit analysis of plates using the EFG method and second-order cone programming," International Journal for Numerical Methods in Engineering 2009;78(13):1532-55.

[15] Le CV, Gilbert M, Askes H, "Limit analysis of plates and slabs using a meshless equilibrium formulation," International Journal for Numerical Methods in Engineering 2010;83(13):1739-58.

[16] Shepard D, "A two-dimentional interpolation function for irregularly-spaced data," Proc. ACM Nat. conf 1968. p.517-24

[17] Gordon W, Wixom J, "Shepard's method of Metric Interpolation to Bivariate and Multivariate Interpolation.," Mathematics of computation 1978;32(141):253-64.

[18] Chen JS, Wu CT, Yoon S, You Y, "A stabilized conforming nodal integration for Galerkin Mesh-free methods," International Journal for Numerical Methods in Engineering 2001;50:435–66.

[19] Davis, E. H. , and Booker, J. R., "The effect of increasing strength with depth on the bearing capacityof clays," Geotechnique, Vol. 23, pp. 551–63,1973.

Int. J. of GEOMATE, June, 2013, Vol. 4, No. 2 (Sl. No. 8), pp. 511-515.
MS No. 2105 received June 13, 2012, and reviewed under GEOMATE publication policies.

Pertinent discussion including authors' closure, if any, will be published in the June 2014 if the discussion is received by Dec., 2013.
Corresponding Author: S.M. Binesh