KONDO THEORY FOR SPHERICAL SHELL TECTONICS

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ABSTRACT: The buckling phenomenon of a flat or spherical shell lithosphere (tectonic plate) has been investigated in previous research. However, these studies do not give information regarding the curvature effect in the buckling phenomenon. Kondo applied Riemannian geometry to the yielding or buckling of curved materials. When the Riemannian manifold (V_n dimensional manifold) with a nonzero Euler–Schouten curvature tensor is manifested in the enveloping manifold (Euclid space: V_m dimensional manifold), the included Riemannian manifold (dimension V_n) protrudes into the enveloping manifold (dimension V_m). The curvature effect for the buckling phenomenon of materials can be formulated by a force-balance equation from mechanics and the Euler–Schouten curvature tensor from differential geometry. In this paper, using the Euler–Schouten curvature tensor from differential geometry, the authors derive a formulation for the buckling phenomenon with the curvature effect for a spherical shell lithosphere as a buckling equation with high-order strain for lithosphere deformation.

1. INTRODUCTION

The mechanism of flat or spherical shell lithosphere deformation is presented by using buckling theory [e.g., 1–4]. Yamaoka et al. [5] pointed out the buckling phenomena of the subducting lithosphere due to the sphericity of the earth. Moreover, Fukao et al. [6] and Yamaoka [7] denoted the similarity of lithosphere buckling with cylindrical buckling of spherical shells through experiments and numerical simulations based on the nonlinear finite element method. Kikuchi and Nagahama [8] found a new linear relationship between the Batdorf parameter and normalized hydrostatic pressure along the bottom circumferential edge of a hemisphere in spherical shell tectonics. The Batdorf parameter for a subducting lithosphere is equivalent to the length of the slab and is also related to the wavelength (length of the island arc) of buckling. However, previous research has not revealed the buckling equation with high-order strain for the curvature of the lithosphere.

The buckling equation with the high-order strain effect has been addressed in the field of engineering science [9], in which Riemannian geometry has been applied to the yielding and buckling of curved material. Kondo used the concept of dimension protrusion, in which buckling in the two dimensions of a flat plate can occur in three-dimensional space (Fig. 1). In general, when the Riemannian manifold of dimension V_n with a nonzero Euler–Schouten curvature tensor exists in the enveloping manifold (Euclid space) of dimension V_m, the included Riemannian manifold of V_n protrudes into the enveloping manifold of dimension V_m. The Euler–Schouten curvature tensor and force-balance equation provide an understanding of material science with regard to the curvature effect for the buckling phenomenon in terms of differential geometry. In this paper, the authors derive the buckling phenomenon with the curvature effect for a spherical shell lithosphere. The authors can apply the Euler–Schouten tensor to the buckling equation with high-order strain for lithosphere deformation. Using deformation theory based on Riemannian space for the buckling system of the flat plate and spherical shell, the equation for the lithosphere deformation and buckling can be derived from the Euler–Schouten curvature tensor. Therefore, this curvature tensor is an important tensor for lithosphere deformation. This paper is an extended paper of the Proceedings of the Seventh International Conference on Geotechnique, Construction Materials and Environment (GEOMATE-Mie 2017) [10].
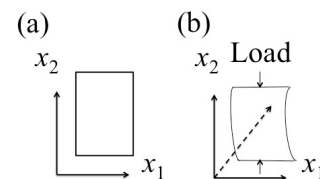

Fig. 1 Two-dimensional pre- and post-buckling of the flat plate. (a) Pre-buckling of material is in two dimensions. (b) Post-buckling of material is in three dimensions.

2. PREVIOUS BUCKLING THEORY

The authors briefly introduce the previous buckling theory used in earth science. The mechanism of geological folding (flat crust buckling) was described by use of buckling theory [1]. First, the authors introduce the balance equation of the flat crust as follows (Fig 2):

$$\frac{dV}{dx} = -q(x), \qquad (1)$$

$$\frac{dM}{dx} = V, \qquad (2)$$

where V is the shearing force, x is the coordinate of the system, $q(x)$ is the load, and M is the moment. Moreover, the authors can derive a relationship between the load and moment using Eq. (1) and Eq. (2),

$$\frac{d^2M}{dx^2} = -q(x). \qquad (3)$$

Then, the authors can use the proportion of curvature to describe the flexure moment and curvature equations,

$$EI\kappa = M, \qquad (4)$$

$$\kappa = -\frac{d^2w}{dx^2}, \qquad (5)$$

where E is Young's modulus, I is the second moment, κ is curvature, and w is deflection. Next, from Eqs. (1)–(5),

$$EI\frac{d^2w}{dx^2} = -M, \qquad (6)$$

$$EI\frac{d^3w}{dx^3} = -V, \qquad (7)$$

$$EI\frac{d^4w}{dx^4} = q(x). \qquad (8)$$

Eq. (8) is the crust buckling equation. If the crust undergoes axial compressive force, the authors can write the equation as follows:

$$EI\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} = q(x), \qquad (9)$$

where P is the axial compressive force.

The mechanism of spherical shell lithosphere deformation is presented, assuming buckling theory [e.g., 2–4]. Turcotte and Schubert [4] showed that lithosphere deformation is given by:

$$D\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} + (\rho_c + \rho_m)ghw = q(x), \quad (10)$$

where D is the shear modulus, ρ_c is crust density, ρ_m is mantle density, g is gravitational acceleration, and h is height (Fig. 3). However, previous studies have not demonstrated a high-order curvature effect in the buckling phenomenon.
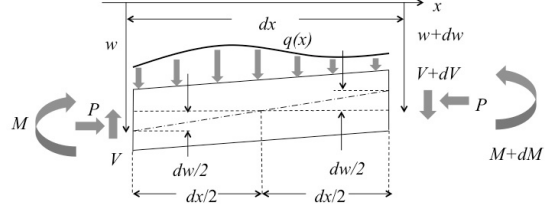

Fig. 2 Buckling of flat crust deformation, where x is the coordinate of the system, $q(x)$ is the load, w is deflection, M is moment, P is axial compressive force, and V is shearing force.
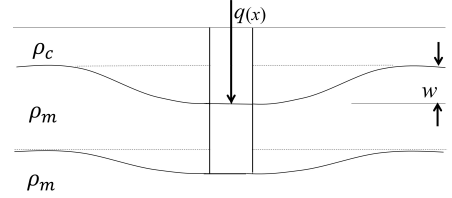

Fig. 3 Buckling of lithosphere deformation, where ρ_c is the crust density, ρ_m is mantle density, w is deflection, and $q(x)$ is the applied load [modified from 4].

3. KONDO THEOREM

In a previous paper [9], Kondo provided some important information regarding materials science. By using differential geometry in the study of the buckling of plate and shell, Kondo [9] proceeded as follows: The object was placed in Cartesian coordinates and m dimensional Riemannian space (envelope space V_n; i,j are frames of reference in Fig. 4).
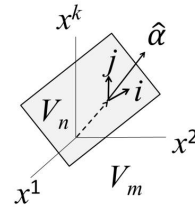

Fig. 4 Riemannian manifold of dimension V_n with a nonzero Euler–Schouten curvature tensor in the enveloping manifold (Euclid space) of V_m, with coordinates $x^{(1,2,\cdots,k)}$, and frames of reference i,j. $\hat{\alpha}$ is the normal vector of V_n.

Moreover, the deformation of the normal direction α of the object can be expressed using strain and Christoffel symbols

$$y^{\alpha}_{;i;j} = O(\varepsilon), \qquad (11)$$

$$\begin{Bmatrix} k \\ i\,j \end{Bmatrix} = O(\varepsilon), \qquad (12)$$

where y is the deflection in V_n space, $\begin{Bmatrix} k \\ i\,j \end{Bmatrix}$ is Christoffel symbol of the second kind, and $O(\varepsilon)$ is order ε of the Landau symbol. In this case, the object deformation is expressed by normal direction ($\hat{\alpha}$):

$$y^{\hat{\alpha}}_{;k} = O(\varepsilon). \qquad (13)$$

Then, the subspace of the enveloping protrusion (V_n to V_m space protrusion) is defined by the Euler–Schouten curvature tensor $H^{\hat{\alpha}}_{.ij}$ [11]:

$$H^{\hat{\alpha}}_{.ij}\left(\equiv y^{\hat{\alpha}}_{;i;j}\right) = \frac{\partial^2 y^{\hat{\alpha}}}{\partial x^i \partial x^j} - y^{\hat{\alpha}}_{;k}\begin{Bmatrix} k \\ i\,j \end{Bmatrix}. \quad (14)$$

The second term of the components of the curvature tensor is $O(\varepsilon^2)$ order. Hence, the authors omitted the second term in the small deformation theory:

$$H^{\hat{\alpha}}_{.ij} \approx \frac{\partial^2 y^{\hat{\alpha}}}{\partial x^i \partial x^j}. \qquad (15)$$

The approximate expression is compared to expression (11). It is configured by omitting a very small amount. Metrics are defined by the following:

$$g_{ij} = \delta_{ij} + O(\varepsilon), \qquad (16)$$

where g_{ij} and δ_{ij} are matrix elements. The authors use the equation of equilibrium of forces and equations of equilibrium with a small strain,

$$(J - F)^i = \frac{\partial \Sigma^{ij}}{\partial x^j}, \qquad (17)$$

$$(L - \varrho)^{\hat{\alpha}} = \frac{\partial^2 G^{\hat{\alpha}ij}}{\partial x^i \partial x^j} + H^{\hat{\alpha}}_{.ij}\Sigma^{ij}, \quad (18)$$

where J is the tangential force per unit m volume, F is the difference in the tangential frictions at the upper and lower boundaries, Σ^{ij} is the stress components in the shell space, L is the normal force per unit m volume, ϱ is the difference in the tangential frictions at the upper and lower boundaries, and $G^{\hat{\alpha}ij}$ is the Euler–Schouten curvature contravariant tensor.

The coordinate transformation law is expressed by:

$$G^{.ij}_{\alpha} = B^{..ijkl}_{\alpha\beta}H^{\beta}_{.kl}, \qquad (19)$$

$$G^{\alpha ij} = \alpha^{\alpha\beta}G^{.ij}_{\beta}, \qquad (20)$$

$$B^{\alpha\cdot ijkl}_{.\beta} = \alpha^{\alpha\gamma}B^{..ijkl}_{\gamma\beta}, \qquad (21)$$

where B is the contravariant tensor in n-space and β and l are indices. From Eq. (15), the balance equations (Eqs. 17–18) and the coordinate transformation (Eqs. 19–21), the authors can write:

$$(L - \varrho)^{\hat{\alpha}} = \frac{\partial^2}{\partial x^i \partial x^j}\left(B^{\hat{\alpha}\cdot ijkl}_{\beta}\frac{\partial^2 y^{\hat{\beta}}}{\partial x^k \partial x^l}\right) \\ + \frac{\partial^2 y^{\hat{\alpha}}}{\partial x^i \partial x^j}\Sigma^{ij}. \quad (22)$$

Moreover, when the material is isotropic, the B^{ijkl} tensor can be expressed with the constant B,

$$B\Delta\Delta y + \Sigma^{ij}\frac{\partial^2 y}{\partial x^i \partial x^j} = L - \varrho. \qquad (23)$$

4. DISCUSSION

The authors consider the relationship between previous research and Kondo theory. DiDonna [12] presented the buckling equation for an elastic sheet,

$$\frac{Eh^3}{12(1-\nu^2)}\frac{d^2}{dx^2}\left(\frac{d^2 w}{dx^2}\right) - \frac{\partial}{\partial x}\left(\sigma_{ij}\frac{\partial w}{\partial x}\right) = P_e, (24)$$

where h is thickness, ν is Poisson's ratio, σ_{ij} is the stress, and P_e is an external pressure field. The curvature tensor can be written as the derivative of a continuous curvature potential f [12]

$$C_{ij} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x}, \qquad (25)$$

where C_{ij} is the curvature tensor. Here, the potential f is not identical to the local function w used above, but is approximately equal to w for nearly flat surfaces [12]. The parameter C_{ij} is the Euler–Schouten curvature tensor. Hence, this buckling equation for an elastic sheet is a low-order strain equation. Furthermore, the authors write σ_{ij} in terms of the stress potential χ

$$\sigma_{ij} = \epsilon_{ik}\epsilon_{jl}\frac{\partial}{\partial x}\frac{\partial \chi}{\partial x}, \qquad (26)$$

where ϵ_{ik} and ϵ_{jl} are antisymmetric tensors. In terms of the potentials χ and f, the von Karman equations [13] can be expressed by

$$D\nabla^4 f = [\chi, f] + P_e, \qquad (27)$$

$$\nabla^4 \chi = -\frac{1}{2}[f, f]. \qquad (28)$$

This von Karman equation is a generalized buckling equation (23) for low-order strains. Moreover, the buckling can be described by a simplification of Eq. (9). It is apparent that Eq. (23) is the extended two-dimensional version equation of Eq. (9) with constant vertical forces (i.e., q = const.) and without body forces. It is also apparent that a two-dimensional version equation (e.g., [4]) is included in the kind of equations reduced from Eq. (23). Thus, various kinds of geological deformations can be described by the simplification of the generalized buckling equation. For example, the mechanism of geological folding occurs by plate motion. In this case, three-dimensional flat plate buckling (geological folding) can be described by Eq. (23), the buckling equation of the flat plate.

The buckling phenomena of the subducting lithosphere due to the sphericity of the earth have been studied in the context of spherical shell tectonics [5–7]. The slab length is approximately proportional to the arc length, and the lithosphere thickness is related to the lithosphere age. Moreover, the length of the deformable portion of the shell corresponds to the length of the subducting slab. The lithosphere is defined by the length of the Wadati–Benioff zone and the thickness of the shell which corresponds to the effective elastic thickness of the lithosphere. From the dataset of geometrical parameters for subducting lithosphere, Kikuchi and Nagahama [8] presented a new linear relationship between the normalized hydrostatic pressure and the Batdorf parameter as the dimension of the shell (i.e., the flatness). Therefore, in this case, Eq. (23) becomes the buckling equation of the spherical shell lithosphere as a three-dimensional Riemannian manifold (V_n).

5. CONCLUSIONS

Buckling of the plate and shell in the field of materials science is considered, using differential geometry [9]. From using the Euler–-Schouten curvature tensor of the shell as a Riemannian manifold and the force-balance equation for the shell, the authors derived a unified theory for buckling of spherical shells. When the Riemannian manifold of dimension V_n with a nonzero Euler–Schouten curvature tensor exists in an enveloping manifold (Euclid space) of dimension V_m, the including Riemannian manifold of V_n protrudes into the enveloping manifold of dimension V_m. From the Euler–Schouten curvature tensor on the shell as a Riemannian manifold and the force-balance equation for the shell, the authors derived a unified theory for buckling of flat plates or spherical shells in Eq. (23).

6. ACKNOWLEDGEMENTS

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