# AN EXACT DISPLACEMENT BASED FINITE ELEMENT MODEL FOR AXIALLY LOADED PILE IN ELASTO-PLASTIC SOIL

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**ABSTRACT:** A displacement based finite element method for analyzing axially loaded pile embedded in finite depth of elasto-plastic soil is presented. The investigation herein is conducted on the condition of shape function by which exact value may be reproduced at the nodal point regarding to a few number of element. The examined shape functions which satisfy the homogeneous governing equations in elastic and plastic soil are introduced to obtain the so-celled exact element stiffness matrix via total potential energy principle. Numerical examples of elasto-static pile, embedded in elasto-plastic Winkler foundation illustrates the accuracy of proposed element compare with conventional finite element shape functions. Axial force and displacement solutions show very good agreement with data from the available literature. Then the proposed shape functions are also used to conduct free vibration analysis of axially loaded pile embedded in elastic soil. The results from finite element modal analysis show fairly accurate compare with analytical solutions.

*Keywords: Axially loaded pile, Displacement method, Finite element, Shape function, Soil-pile interaction*

## 1. INTRODUCTION

Accuracy of a finite element solution depends significantly on the extent to which the assumed displacement pattern is able to reproduce the actual deformation of the structure. For a one-dimensional problem with single variable, Tong [1] proved that the finite element nodal solution can be obtained exactly if the assumed shape functions satisfy the homogeneous differential equation of the problem. Kanok-Nukulchai *et al*. [2] extended Tong's concept to one-dimensional problem with several dependent variables, i.e., authors developed an exact two-node deep beam element. Ma [3] also applied concept of nodal exact shape function to solved axial vibration problems of elastic bars, exact solution was obtained for undamped harmonic response analysis. Force-based formulation of pile-embedded in elastic soil was proposed in Reference [4] and obtained the exact solution for problem taken from Li *et al*. [5]. The displacement-based formulation was also proposed in Reference [6], the exact nodal-displacements and forces was obtained for elastic soil.

In this study, the nodal exact shape function concept suggested by Tong [1] and Buchart [6] is employed to solve the elastic bar embedded in elasto-plastic soil problem. Two sets of shape functions which satisfies the homogeneous differential equation will be derived for developing a present bar element. The stiffness matrix and nodal force vector are expressed based on total potential energy principle. Example of elasto-static pile embedded in elasto-plastic soil is solved to verify the accuracy and efficiency of proposed bar element. Free vibration of a pile embedded in elastic soil foundation using the consistent mass matrix for this pile element is also investigated.

## 2. MATHEMATICAL FORMULATION

### 2.1 Problem Definition

The analysis considers a single circular pile, with diameter $d$ (cross section $A = 0.25\pi d^2$ and perimeter $U = \pi d$), embedded into soil deposit (Fig. 1). The pile has a total length $L$ and is subjected to an axial force $P_0$ at the pile head which is flush with the ground surface. The soil medium is assumed to be elastio-plastic, isotropic and homogeneous, with elastic properties described by equivalent spring coefficient $k_s$. Once the soil displacement go beyond the yielding displacement $w_*$, the shear resistance will keeps constant as long as the displacement increases. The soil bearing capacity at pile's end is presented by coefficient $k_b$. The pile is assumed to behave as an elastic column with Young's modulus $E$. The Poisson's ratio of the pile material is neglected. Figure 1(a) shows the elastic and plastic zone occurred in soil due to the axial displacement $w_0$ at top pile head is greater than yielding displacement $w_*$ occurred at depth $z_0$ from pile head. Hence, the length of an elastic portion in Fig. 1 is denoted by $\ell$, the magnitude of axial displacement in elastic portion is less than or equal to yielding displacement.

**2.2 Governing Differential Equations**

Consider one-dimensional element in Fig. 2, the total potential energy of this soil-pile element subjected to the axial forces $P_1$ and $P_2$ is defined as
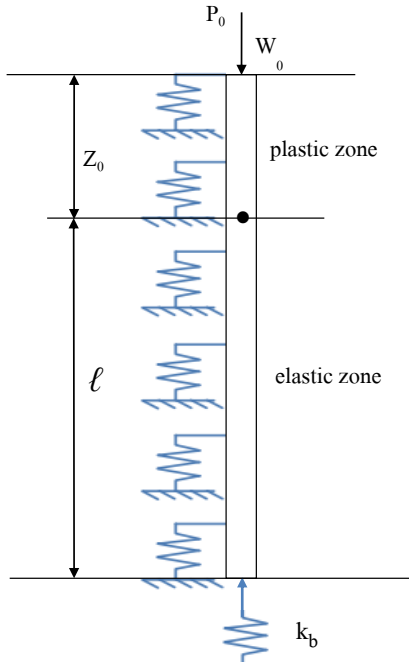


(a)
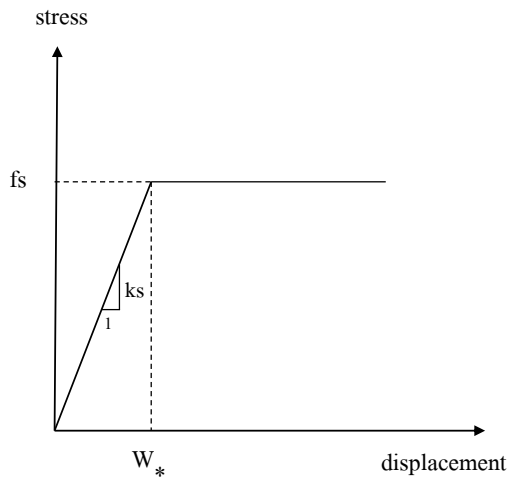


(b)

Fig. 1    Axially loaded pile and soil model

the sum of internal potential energy (strain energy) and the external potential energy due to external load as follow:

$$\Pi = \frac{1}{2}\left[ EA \int_0^L \left(\frac{dw}{dz}\right)^2 dz + k_s U \int_0^L w^2 dz \right] - P_1 w_1 - P_2 w_2 \tag{1}$$

where $w(z)$ is the vertical pile displacement at depth $z$. The first and second terms in Eq. (1) represent the strain energy in pile and surrounding soil, respectively.
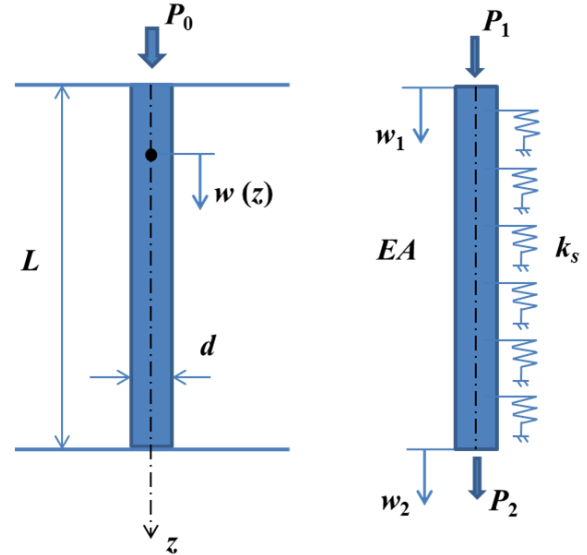


Fig. 2    Axially loaded pile embedded in soil and corresponded two-node finite element model.

The first variation of Eq. (1) leads to

$$\delta\Pi = EA \int_0^L \left(\frac{d\delta w}{dz}\right)\left(\frac{dw}{dz}\right)dz + k_s U \int_0^L (\delta w)w\, dz - P_1 \delta w_1 - P_2 \delta w_2 \tag{2}$$

Applying the appropriate Gauss-Green theorem to Eq. (2) and setting $\delta\Pi = 0$, gives the differential equation for equilibrium

$$-EA \frac{d^2 w}{dz^2} + k_s U w = 0 \quad \text{for} \quad 0 < z < L \tag{3}$$

and a set of natural boundary conditions as follows

$$P_1 = -EA \left.\frac{dw}{dz}\right|_{z=0} \quad \text{and} \quad P_2 = EA \left.\frac{dw}{dz}\right|_{z=L} \tag{4}$$

**2.3 Shape Functions for An Exact Pile Element**

Following the concept presented by [1] and [2], a pile element is developed with the shape functions that satisfy the homogeneous differential equation, Eq. (3). Two groups of shape functions are derived following elastic and plastic soil conditions.

### 2.3.1 Elastic soil conditions

The field variable $w$, which satisfy Eq. (3), can be represented by the following hyperbolic function

$$w(z) = c_1 \cosh(\alpha z) + c_2 \sinh(\alpha z) \tag{5}$$

where $\alpha^2 = k_s U / EA$ is the characteristic parameter of pile. By applying the nodal displacement boundary conditions to elastic soil portion (Fig. 1), Eq. (5) can now be expressed in terms of nodal variables, $w_1$ and $w_2$, as follows

$$w(z) = w_1 N_1(z) + w_2 N_2(z) \tag{6}$$

where the shape functions can be expressed as

$$N_1(z) = \frac{\sinh[\alpha(\ell - z)]}{\sinh \beta} \tag{7}$$

$$N_2(z) = \frac{\sinh(\alpha z)}{\sinh \beta} \tag{8}$$

with the non-dimensional parameter $\beta = \alpha^\ell$.

### 2.3.2 Plastic soil conditions

Consider the plastic soil portion, upper portion in Fig. 1, displacement $w$ in second term of equilibrium equation (3) have to be replaced by yielding displacement $w_*$ as follow

$$-EA \frac{d^2 w}{dz^2} + k_s U w_* = 0 \quad \text{for} \quad 0 < z < z_0 \tag{9}$$

The field variable $w$, which satisfy homogeneous differential equation part of Eq. (9), can be represented by linear function of soil depth. Hence, displacement shape function of plastic portion can be expressed as a ramp function used in typical linear FEM, *i.e.*

$$N_1(z) = \frac{(z_0 - z)}{z_0} \tag{10}$$

$$N_2(z) = \frac{z}{z_0} \tag{11}$$

## 2.4 Derivation of Element Stiffness Matrices

Refer to first variation of strain energy terms in Eq. (2), due to the arbitrariness of $\delta w$ (see also [7]), the element stiffness matrix for pile and soil can be expressed as follows:

$$K_{ij}^{pile} = EA \int_0^L \left( \frac{dN_i}{dz} \right) \left( \frac{dN_j}{dz} \right) dz \tag{12}$$

$$K_{ij}^{soil} = k_s U \int_0^L N_i N_j \, dz \tag{13}$$

where the indices $i$ and $j$ are ranged from 1 to 2.

### 2.4.1 Element stiffness for an elastic soil portion

Substituting shape functions from Eq. (7) and (8) into element stiffness formulation in Eq. (12) and (13), the component of element stiffness matrices for elastic soil portion can be explicitly expressed as

$$K_{11}^{pile} = K_{22}^{pile} = \frac{\alpha EA}{2} \left( \beta \operatorname{csch}^2 \beta + \coth \beta \right)$$

$$K_{12}^{pile} = K_{21}^{pile} = -\frac{\alpha EA}{2} \operatorname{csch} \beta \left( 1 + \beta \coth \beta \right) \tag{14}$$

and

$$K_{11}^{soil} = K_{22}^{soil} = \frac{k_s U}{2\alpha} \left( \coth \beta - \beta \operatorname{csch}^2 \beta \right)$$

$$K_{12}^{soil} = K_{21}^{soil} = -\frac{k_s U}{2\alpha} \operatorname{csch} \beta \left( 1 - \beta \coth \beta \right) \tag{15}$$

### 2.4.2 Element stiffness for plastic soil portion

Substituting shape functions from Eq. (10) and (11) into element stiffness formulation in Eq. (12) and (13), the component of element stiffness matrices for plastic soil portion can be explicitly expressed as

$$K_{11}^{pile} = K_{22}^{pile} = -K_{12}^{pile} = -K_{21}^{pile} = \frac{EA}{z_0} \tag{16}$$

and

$$K_{11}^{soil} = K_{22}^{soil} = 2K_{12}^{soil} = 2K_{21}^{soil} = \frac{k_s U z_0}{3} \tag{17}$$

## 2.5 Consistent Mass Matrix

In free vibration analysis, the consistent mass matrix has to be constructed. Formula of consistent mass matrix is similar with soil stiffness matrix in Eq. (15) or (17), except the factor $k_sU$ replaced by $\rho A$, i.e.

$$M_{ij} = \rho A \int_0^L N_i N_j \, dz \qquad (18)$$

where $\rho$ = mass density of pile material. An explicit expression of this consistent mass matrix will be omitted for the sake of simplicity and clarity of the presentation.

## 3. NUMERICAL EXAMPLES

In this section, two numerical examples are presented to illustrate the effectiveness of the finite element proposed in the previous section. Application of this pile element to the free vibration analysis is also demonstrated by the second example. Analytical solutions of all problem tests are available in the literatures [6, 8, 9, and 10].

## 3.1 Static Analysis

A bored pile was installed in the medium silt clay and the end bearing layer is sandstone. The pile length is 45 m, and the diameter $d$ = 1 m. The elastic modulus of pile shaft $E = 2.2 \times 10^7$ kPa. From soil tests, the values of equivalent soil elastic coefficient $k_s$ = 12000 kPa/m, the yielding displacement of soil $w_* = 2.6$ mm, and end bearing stiffness $k_b$ = 684000 kPa/m [8]. The value $k_bA$ is added into the last diagonal member of stiffness matrix.

The numerical test was performed using two element assembly to construct three algebraic equations. Assuming the value of plastic depth $z_0$, the displacement at bottom end was solved from third row of algebraic equation. Then, the displacement at pile head $w_0$ was obtained from second row of algebraic equation, and the value of load $P_0$ at pile head was computed. Two cases were run to compare the results: (i) proposed stiffness, Eqs. (14) and (15) were used to constructed element stiffness of bottom part, and Eqs. (16) and (17) for upper part, and (ii) two conventional linear elements [7] were used to constructed the global stiffness matrix.

The results from two cases are shown in Table 1. As expect, the present pile element model prove to be flawless: finite element solutions of proposed element are identical to the exact analytical solutions in reference [8]. The conventional linear element behaves stiffer than proposed element and exact solutions, greater pile head load is required to obtain yield displacement at pile head.

Table 1 Calculated loads and settlement of the pile at any values of plastic depth.

| $z_0$ (m) | Present (Exact) | | Linear FEM | |
|---|---|---|---|---|
| | $P_0$ (kN) | $w_0$ (mm) | $P_0$ (kN) | $w_0$ (mm) |
| 0 | 2086 | 2.60 | 2451 | 2.60 |
| 9 | 2951 | 3.91 | 3192 | 4.03 |
| 18 | 3796 | 5.64 | 3931 | 5.78 |
| 27 | 4593 | 7.71 | 4648 | 7.80 |
| 36 | 5291 | 9.95 | 5302 | 9.97 |
| 45 | 5807 | 11.98 | 5807 | 11.98 |

## 3.2 Free Vibration Analysis

Natural frequencies and mode shapes of a fixed-ended pile is computed with ten pile elements. A bored pile was installed in the medium clay of length 50 m and diameter $d$ = 1 m. Equivalent soil elastic coefficient $k_s$ = 34200 kPa/m. The end bearing stiffness is very large (fixed at the bottom). The unit mass is of pile shaft is taken as $\rho$ = 2400 kg/m$^3$. To perform modal analysis, element stiffness and consistent mass matrices of proposed element and linear conventional element are constructed with assumed elastic soil condition. Result of the natural frequencies, compared with the exact theory [10] is shown in Table 2. Percent error of natural frequencies, which are obtained from conventional linear element are slightly better than present element. The first three mode shapes of proposed element are also plotted in Fig. 3 and appear to be almost indifferent from the exact ones.

Table 2 Circular frequency of fixed-ended pile

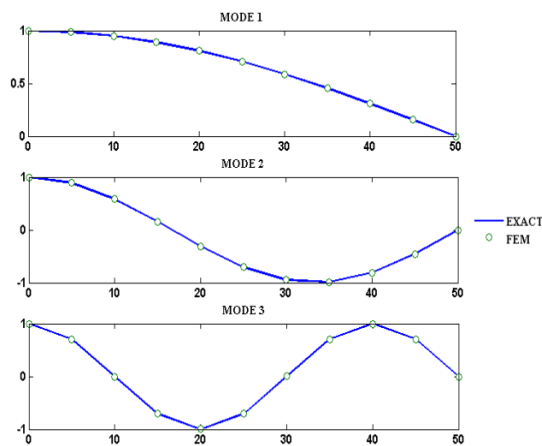| Mode | Circular Frequency (rad/sec) | | |
|---|---|---|---|
| | Ten piles (Present) | Ten piles (Linear) | Exact Theory |
| 1 | 153 | 153 | 153 |
| 2 | 313 | 312 | 309 |
| 3 | 504 | 502 | 490 |
| 4 | 712 | 710 | 676 |
| 5 | 938 | 935 | 864 |
| 6 | 1184 | 1181 | 1053 |
| 7 | 1446 | 1443 | 1242 |
| 8 | 1711 | 1708 | 1432 |
| 9 | 1943 | 1940 | 1621 |
| 10 | 2084 | 2082 | 1811 |

Fig. 3   Fixed tip pile problem: the first three mode shapes from the free vibration analysis of a ten-element model.

## 4. CONCLUSIONS

The new finite element model for pile subjected to axial load is proposed. A necessary condition for the present finite element model to reproduce the exact values at the nodal points is that it has to satisfy the homogeneous differential equation of the problem. Numerical example for static load pile embedded in elasto-plastic soil indicates that an exact finite element solution can be obtained even with minimum number of element used. In addition, the same shape functions can produce fairy satisfactory results in free vibration problem on three fundamental modes.

## 5. REFERENCES

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## APPENDIX

### A.1 Exact Solution of Pile Static Load Test

Prescribe the boundary conditions (4) into the solution of governing equations (3) and (9). The load applied to the pile top could be expressed as

$$P_0 = w_* \left[ \alpha\, EA \tanh \left( \alpha^\ell + \gamma \right) + z_0 k_s U \right] \qquad (19)$$

and the displacement at the pile top is therefore

$$w_0 = w_* + \frac{1}{2} w_* \left[ \tanh \left( \alpha^\ell + \gamma \right) + \frac{z_0 k_s U}{\alpha\, EA} \right]^2$$
$$- \frac{1}{2} w_* \tanh^2 \left( \alpha^\ell + \gamma \right) \qquad (20)$$

The variable $\gamma$ is the characteristic value of the end bearing capacity of soil at pile tip.

$$\tanh \left( \gamma \right) = \frac{k_b}{\alpha\, E} \qquad (21)$$

In derivation of Eqs. (19) and (20), the soil of end bearing capacity is assumed in the elastic condition. Detail derivation of Eqs. (19)–(21) are described in reference [8].

### A.2 Free Vibration of Fixed-Ended Pile

The governing equation for free vibration of pile is expressed as

$$-EA\frac{\partial^2 w}{\partial z^2} + k_s U w = \rho A \frac{\partial^2 w}{\partial t^2}, \ 0 < z < L \qquad (22)$$

Assume that pile subjected to fixed boundary condition at the pile tip $w = 0$ at $z = L$. Natural circular frequency $\omega$ and mode shape $\psi$ of pile embedded in soil can be expressed as

$$\psi_n = \cos(\lambda_n z); \ n = 1, 2, 3, \ldots \qquad (23)$$

$$\omega_n = \sqrt{\frac{(\lambda_n^2 + \alpha^2) E}{\rho}}; \ n = 1, 2, 3, \ldots \qquad (24)$$

where the wave number parameters $\lambda_n$ are defined as

$$\lambda_n = \left(n - \frac{1}{2}\right)\frac{\pi}{L}; \ n = 1, 2, 3, \ldots \qquad (25)$$

Note that Eq. (23) always satisfies boundary condition at pile tip, *i.e.* $\psi_n(L)$ always equal to zero.

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