NUMERICAL ANALYSIS ON THE OCCURRENCE OF THERMAL CONVECTION IN FLOWING SHALLOW GROUNDWATER

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ABSTRACT: The occurrence of thermal convection in groundwater could significantly change groundwater flows, and this phenomena could also alter solute transport. In the case of a shallow groundwater, thermal convection might affect the surface water quality because it directly interacts with surface waters such as lakes and marshes. Hence, understanding the conditions that induce thermal convection is important for groundwater management. In this study, the conditions for the onset and decay of thermal convection in a flowing shallow groundwater are investigated by conducting numerical experiments. Through the computational analysis using the attractor reconstruction technique, three flow regime types were found, showing that not only the Rayleigh number, which is a well-known influential factor, but also the groundwater flow rate affects the flow regime.

Keywords: Density-Driven Flow, Rayleigh Number, Bifurcation, Fixed Point, Limit Cycle

1. INTRODUCTION

In the field of geothermal dynamics, large-scale spatial and temporal phenomena in groundwater such as thermal convection and circulation rooted in magmatic heat sources have been a primary area of research interest. Typically, the objective domain ranges from a few kilometers to tens of kilometers, and the time scale ranges from hundreds to thousands of years [1], [2]. Recently, the thermal dynamics in relatively shallow groundwater has been actively investigated for several reasons. One reason is that the temperature distribution in shallow groundwater could be an important clue for understanding the subsurface structure and groundwater flow [3]. Another reason is that geothermal heat pump technology has received attention as one possible alternative for clean energy [4]. Generally, thermal convection is rarely considered in shallow groundwater except for cases around heat sources such as hot springs. However, thermal convection on the scale of weeks or months could occur under some conditions such as groundwater under snow coverage and the intrusion of snowmelt water into groundwater [5]. Tortuous groundwater flows by density-driven flow could drastically change the solute and heat transport, so it is essential to understand the onset, growth, and decay of the density-driven flow for groundwater management. When a certain physical condition varies continuously, the occurrence of such qualitative changes in the groundwater flow regime is considered a bifurcation in the nonlinear dynamics [6], and hydrodynamic stability has been one of the central problems of fluid mechanics [7].

In this study, two cases that could cause thermal convection are targeted: snow coverage and cold water intrusion. This study is aimed at identifying conditions for the occurrence of thermal convection under those two situations through systematic numerical experiments and subsequent analyses.

2. MODEL DESCRIPTION 2.1 Governing Equations

A vertical two-dimensional coordinate system, where x and z are the horizontal and vertically upward directions, respectively, is used in this study. To reduce the number of parameters included in the governing equations, a dimensionless version is employed. The majority of the dimensionless equation derivations for groundwater flow and thermal transport are based on Holzbecher [6] and Kawabata et al. [8].

The final version of the governing equation for groundwater flow with variable water density is described with the stream function under the following assumptions: the incompressibility of water, rigidity of the aquifer, and absence of sources or sinks. The Boussinesq approximation is also employed.

$$\frac{\partial}{\partial X} \left(\frac{\mu^r}{\kappa_z^r} \frac{\partial \Psi}{\partial X} \right) + \frac{\partial}{\partial Z} \left(\frac{\mu^r}{\kappa_x^r} \frac{\partial \Psi}{\partial Z} \right) = -Ra \frac{\partial \theta}{\partial X} \qquad (1)$$

$$Ra = \frac{\gamma Hg \kappa_0 \Delta \rho}{\mu_0 D}, \qquad (2)$$

$$(U,W)^{T} = \left(\frac{\partial \Psi}{\partial Z}, -\frac{\partial \Psi}{\partial X}\right)^{T},$$
 (3)

$$\theta = \frac{T - T_{\min}}{T_{\max} - T_{\min}},\tag{4}$$

$$\rho(\theta) = \rho(T_{\min}) - \Delta \rho \theta , \qquad (5)$$

$$\mu(\theta) = \mu^{r}(\theta)\mu_{0}, \qquad (6)$$

$$\kappa_x = \kappa_x^r \kappa_0, \tag{7}$$

$$\boldsymbol{\kappa}_{z} = \boldsymbol{\kappa}_{z}^{r} \boldsymbol{\kappa}_{0} \tag{8}$$

Here, X and Z are the dimensionless coordinates for x and z, respectively; Ψ is the dimensionless stream function; Ra is the Rayleigh number; ρ is the water density, which is assumed to be proportional to the dimensionless temperature θ ; μ is the viscosity of water; μ^r is a relative value based on the representative viscosity μ_0 ; κ_x and κ_z are the intrinsic permeability in the x and z directions, respectively; κ_x^r and κ_z^r are relative values based on the representative intrinsic permeability κ_0 ; H is the representative length, which is the height of the objective domain in this study; g is gravitational acceleration; γ is the ratio between water and bulk thermal capacities; D is the thermal diffusivity; U and W are dimensionless velocities in the Xand Ζ directions, respectively; T is the temperature; and $T_{\rm max}$ and $T_{\rm min}$ are the maximum and minimum temperatures in the domain, respectively.

The dimensionless version of the governing equation for thermal transport is described as follows:

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial X} \left(\frac{\partial \theta}{\partial X} \right) + \frac{\partial}{\partial Z} \left(\frac{\partial \theta}{\partial Z} \right) - \left(U \frac{\partial \theta}{\partial X} + W \frac{\partial \theta}{\partial Z} \right)$$
(9)

where τ is dimensionless time.

In addition to the governing equations (1) and (9), auxiliary boundary and initial conditions are imposed for the dimensionless stream function and temperature. In this study, a rectangular domain in which groundwater flows horizontally in the absence of thermal convections is considered. Cold water intrudes into the aquifer from part of the top of the domain Γ_2 as illustrated in Fig. 1. For both the stream function and temperature, Dirichlet-type boundary conditions are given to the top and bottom of the domain, and Neumann-type boundary conditions in which $\partial \Psi / \partial v = 0$ and $\partial \theta / \partial v = 0$, where \boldsymbol{v} is the outward normal unit vector to the boundary, are given to the sides of the domain (Fig. 1). The values of the stream function Ψ_{top} and $\Psi_{top} + \Psi_{ent}$ are constant on Γ_1 and Γ_3 , respectively, except for the cold water intruding zone Γ_2 , where the value changes linearly from Ψ_{top} to $\Psi_{top} + \Psi_{ent}$. With respect to the temperature, $\theta_{\rm ent}$ is given to the water intruding zone Γ_2 , and θ_{top} is given to Γ_1 and Γ_3 . Constant values of Ψ_{btm} and θ_{btm} are given to the stream function and temperature, respectively, at the bottom boundary

For the stream function's initial condition, a horizontal flow condition is given where the stream function changes linearly from Ψ_{bm} to Ψ_{top} . Regarding the temperature, the initial conditions linearly change from θ_{bm} to θ_{top} with some

perturbation, as illustrated in Fig. 2. When the Rayleigh number is smaller than $4\pi^2$, which is the critical number for the onset of the Benard convection [5], a relatively large perturbation is given to confirm convergence to a stable fixed point. On the other hand, when the Rayleigh number is greater than or equal to $4\pi^2$, a relatively small perturbation is given to avoid persistence to an unstable fixed point.

2.2 Numerical Model

The standard Galerkin finite element method is employed for spatial discretization of governing equations (1) and (9), and the Crank-Nicolson method is used for the time evolution of governing equation (9), in which the lumped mass matrix approximation is used. The validity of the numerical model was confirmed [8] by the Benard convection and by the Henry and Elder problems, which are standard benchmark problems for density-driven flow [9]. When the flow velocity is large compared with the diffusivity, finer computational meshes are used to limit the maximum local Peclet number in the domain to unity.

3. NUMERICAL EXPERIMENTS 3.1 Computational Settings

To investigate the onset of instability and the subsequent growth and decay, numerical experiments are conducted under various physical conditions regarding the Rayleigh number and basal



Fig. 2 Initial conditions for temperature

groundwater flow velocity. Here, two cases are considered: groundwater flow without cold water intrusion, which is referred to as Case 1, and cold water intrusion, which is referred to as Case 2. Case 1 represents snow coverage on saturated ground, and Case 2 represents waterlogging for soil puddling in spring. The Rayleigh number *Ra* ranges from 0 to 100, and Ψ_{top} ranges from 0 to 20. As the combination of these two variables varies, simulations are conducted to investigate whether instabilities such as thermal convection and fluctuation occur. For Case 1, Ψ_{ent} is set to 0; and θ_{btm} , θ_{top} , and θ_{ent} are set to 1.0, 0.0, and 0.0, respectively. For Case 2, Ψ_{ent} is set to 2; and θ_{btm} , θ_{top} , and θ_{ent} are set to 1.0, 0.7, and 0.0, respectively.

The domain is assumed to be homogeneous and isotropic, so the relative values κ_x^r and κ_z^r for the intrinsic permeability are unity. The relative value μ^r for the viscosity is determined from the following functional relationship to temperature in Kelvin [5]:

$\mu(T) = 10^{-3} \left\{ 1 + 0.015512 \left(T - 293.15 \right) \right\}.$ (10)

3.2 Analysis on Computed Solutions

To quantitatively evaluate whether the stream function and/or the temperature at a certain point fluctuate, moving variance is calculated on each node in the two sampling zones, the cold water intrusion zone and groundwater outflow zone (Fig. 1). The number of time steps used to obtain the variance is 400, which is sufficiently long compared with the cycle length obtained from our preliminary simulations. When the solution converges to a steady state, the variance decreases, and asympotically becomes zero. In contrast, if the solution fluctuates, the variance does not converge to zero.

In addition to the moving variance, the behaviors of the stream function and temperature in a node are analyzed with the attractor reconstruction, which is a non-linear-data analysis technique, and the time series data is transformed to a trajectory in 2-D phase space [10]. In attractor reconstruction,







Fig. 4 Stream functions in Case 1 ($\tau = 4$)



Fig. 5 Trajectories of temperature in phase space in Case 1



Fig. 6 Temporal changes of temperature in Case 1 (Ra = 50)

the coordinates of each point in the trajectory are generated from time series data as $(\chi(\tau), \chi(\tau + \tau_{\rm D}))$ with time delay $\tau_{\rm D}$, where $\chi(\tau)$ is the time series data of the stream function or temperature in the node. If the time series data oscillates sinusoidally, the trajectory becomes circular when the time delay is one-quarter cycle.

3.3 Computed Results

First, the computed results for Case 1 are discussed. The temperature distributions for $\tau = 4$ are shown in Fig. 3, and the corresponding stream functions are shown in Fig. 4. For the condition in which $\Psi_{top} = 0$ corresponds to that for the Benard convection, thermal convection occurs if the Rayleigh number is greater than $4\pi^2$, or 40 and 50 in this case. As shown in Fig. 4, weak circulations occur even in the condition where Ra = 30 and



Fig. 7 Classification of computed conditions in Case 1

 $\Psi_{top} = 0$. The obtained result is considered to be converging to the steady state. For a horizontal groundwater flow, the waves caused by thermal convection are carried by the horizontal flow, and new waves continually arise at the upstream end. A larger Rayleigh number is required to maintain thermal convection as the basal groundwater flow (Ψ_{top}) increases. Figure 5 shows the trajectories of the temperature on the node where the moving variance is maximal in the groundwater outflow zone under each condition. Three trajectory types are found: convergence to a point without a spiral, e.g., the case where Ra = 50 and $\Psi_{top} = 0$; convergence to a point with a spiral, e.g., the case where Ra = 50 and $\Psi_{top} = 16$; and asymptotic approach to a limit cycle with a spiral, e.g., the case where Ra = 50 and $\Psi_{top} = 8$. These three flow regimes are referred to as Type1, Type 2, and Type 3, respectively. Figure 6 shows the temporal change



Fig. 9 Stream functions in Case 2 ($\tau = 4$)

of temperatures of the three above cases. The temporally fluctuating graphs of $\Psi_{top} = 8$ and $\Psi_{top} = 16$ correspond to the spirals in Fig. 5. These qualitative trajectory transformations imply bifurcations. In the Ra = 30 case, which is under the critical value $4\pi^2$, the stable fixed point in $\Psi_{top} = 0$ transforms to a stable spiral when Ψ_{top} is non-zero. For the Ra = 40 and 50 cases, which are over the critical value, the stable fixed point transforms to an unstable spiral, and a limit cycle is formed when Ψ_{top} has a small non-zero value that does not exceed a threshold. If Ψ_{top} has a value beyond the threshold the unstable spiral transforms to a stable spiral, and the limit cycle disappears. The threshold depends on the Rayleigh number. Figure 7 shows the classification of each computed condition into these three types.

Second, the obtained results of Case 2, cold water intrusion into flowing groundwater, are discussed in the same way as Case 1. Figures 8 and 9 show the temperature and stream function distributions of typical patterns after a sufficient time in Case 2, respectively. From these figures, two flow regime patterns are found: steady and unsteady flow regimes. The steady regimes are found when the Rayleigh number is smaller than or equal to 40, or when the basal groundwater flow is large, even with higher Rayleigh numbers. Under such conditions, Fig. 10 shows trajectories

converging to a point without a spiral (Type 1) and with a spiral (Type 2). The unsteady flow regimes in which waves are carried by the basal groundwater flow are found in the remaining conditions. In addition, different fluctuation types are found in the cold water intrusion zone and the groundwater outflow zone as shown in Figs. 10 and 11. For instance, in the cases of Ra = 50 and $\Psi_{top} = 8$, the trajectory approaches a single loop, which is traced in the cold water intrusion zone (Fig. 10 (a)), while the trajectory traces a double loop in the groundwater outflow zone (Fig. 10 (b)). Figure 11 also shows that the temporal change in the cold water intrusion zone is a periodic function that has a single peak in one cycle and that the groundwater outflow zone is a periodic function that has two peaks in one cycle. The classification of the computed conditions is shown in Fig. 12. The flow regimes are found to have changed from Type 3 to Type 2 as Ψ_{top} increased, which is similar to Case 1.

4. CONCLUSION

In this study, the conditions for the onset and decay of thermal convection induced by temperature differences between the top and bottom of the domain and cold water intrusion from the ground surface in shallow groundwater were investigated







Fig. 11 Temporal changes of temperature in Case 2 (Ra = 50, $\Psi_{top} = 8$)



Fig. 12 Classification of computed conditions in Case 2

through numerical experiments. The computed results were analyzed with attractor reconstruction, which visualizes the time series data in a 2-D phase space as a trajectory. From the obtained trajectories, three types of behavioral patterns were found: convergence to a point without a spiral, convergence to a point with a spiral, and asymptotical approach toward a loop. These correspond to a stable fixed point, a stable spiral, and a limit cycle, respectively, and the transformation from one state to another corresponds to a bifurcation when some parameter (condition) varies continuously. In this study, the flow regime changes from a stable spiral to a limit cycle when the Rayleigh number increases, and it changes from a limit cycle to a stable spiral when the basal groundwater flow rate increases. From these results, we can conclude that groundwater flow rate is an important factor that determines the flow regime as well as the Rayleigh number.

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