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ANALYSIS OF STEEL FRAME WITH SEMI-RIGID CONNECTIONS AND CONSTRAINTS USING A CONDENSED FINITE ELEMENT FORMULATION

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ABSTRACT: This paper proposes a novel method for analyzing steel frames with semi-rigid connections and displacement constraints, using a condensed finite element formulation. The paper outlines a systematic approach for constructing the condensed stiffness matrix and the condensed force vector. The proposed analysis can predict accurately the nonlinear effects of connections on the behavior and strength of semi-rigid steel frames. Kishi-Chen power model is applied to describe the nonlinear behavior of semi-rigid connections. The proposed method's effectiveness is demonstrated through analytical and numerical analysis of a semi-rigid steel frame structure, highlighting the procedure's simplicity and robustness. To verify the validity and applicability of the proposed method, several examples of realistic structures and benchmark cases are provided. The strengths predicted by the proposed method demonstrate good agreement with the available experimental results.

Keywords: Finite Element Method, Semi-Rigid Connection, Displacement Constraint, Transformation Technique, Condensed Formulation

1. INTRODUCTION

The analysis and design of structures heavily rely on the accurate modeling of connections. However, this procedure can be challenging in certain cases. Connections in steel structures such as bolting and riveted joints, or after-cracked joints in concrete structures exhibit significant self-deformations that cannot be assumed to be either hinged or rigid. Failing to properly account for connection behavior may lead to an incorrect estimation of the structure's overall behavior. It is generally more appropriate to model connections as semi-rigid with finite stiffness. Additionally, other factors such as the size effect of the column-to-beam connection and displacement constraints at the intersections of structural members cannot always be ignored during the modeling process.

The work of Wilson and Moore [1] was one of the earliest investigations into semi-rigid connections, specifically focusing on determining the rigidity of riveted joints in steel-framed structures. The study tested the rigidity of various joint types, revealing that most of the joints under analyzed stresses could not be considered entirely rigid. Other significant works, such as those conducted by Young and Jack [2] and Rathbun [3], examined the relationship between the moment and relative rotation of welded and riveted connections. The experimental data gathered from these studies served as crucial references for subsequent investigations into semi-rigid connections.

In subsequent decades, data banks for semi-rigid connections expanded, encompassing not only steel structures but also concrete structures in after-crack phases, with a focus on both analytical and experimental subjects. Frye and Morris [4] developed a polynomial model that employs an odd-term polynomial to obtain real moment-rotation curves of steel connections, producing reasonable agreement with experimental data, but generating negative stiffness components that lack physical meaning. Paulay et al. [5] analyzed in detail the behavior of interior beam-column joints in concrete structures subjected to seismic actions and introduced straightforward analytical models of the behavior. Stelmack [6] experimentally verified the validity of analytical models for the response of flexibly connected steel frames. Nethercot [7] reviewed and summarized test data to assess the behavior of steel beam-to-column connections.

Lui and Chen [8] presented one of the most significant works in the development and application of analytical models for finite element analysis regarding semi-rigid connections. Their elasto-plastic model assumed plasticity at connection locations and linear elastic behavior in the remaining member, with beam elements discretized into sub-elements. They used an exponential function to capture the moment-rotation curve of the connection, although their constitutive model was limited to monotonic loading. Kishi and Chen [9] systematically developed a steel connection data bank program, building moment-rotation characteristics for each connection type. In a subsequent work, Kishi and Chen [10] improved Lui and Chen's exponential model and introduced a power model for connections with angles using three parameters: initial elastic stiffness, ultimate moment, and shape parameter. This straightforward procedure enables the evaluation of initial elastic stiffness and the ultimate moment capacity of semi-rigid connections.

The research on semi-rigid connections has expanded to address a variety of structural problems. Stelmack et al. [11] performed cyclic loading tests on a one-bay, two-story semi-rigid steel frame without using a macro-element, instead introducing separate spring elements at the cost of increased nodes and degrees of freedom. Abolmaali [12] proposed separate spring elements to address nonlinear dynamics in steel frames. Chan and Chui [13] established numerical techniques for large deflection and elasto-plastic analysis of steel frames subjected to static and dynamic loads. Vu [14] presented a method for stability analysis of steel frames with semi-rigid connections and rigid zones by using the P-Delta effect. De Lima et al. [15] proposed using neural networks to determine the initial stiffness of beam-to-column joints. Nguyen and Kim [16] developed an analysis method for three-dimensional semi-rigid steel frames that considers nonlinear sources and second-order effects. Macro-frame element models were developed by Saritas and Koseoglu [17] and by Thai and Kim [18] to analyze steel-framed structures with the spread of inelastic behavior along the element length and incorporate nonlinear semi-rigid connections. Truong et al. [19] investigated the use of machine learning methods to estimate the load-carrying capacity of semi-rigidly connected steel structures.

The field of connection modeling remains attractive for two primary reasons: the varied behaviors of connections and the complexity of additional work required in practice. When finite element analysis includes semi-rigid connections, it may be necessary to introduce additional nodes, degrees of freedom, or complex finite element formulations. In practice, explicit element stiffness matrices and force vectors are often used for a given structure with semi-rigid connections, resulting in bulky formulations. This complexity is further compounded when considering other factors in structural analysis, such as multi-component semi-rigid connections, stiff regions, displacement constraints, or diagonal bracing. Therefore, it is crucial to have a simple and systematic approach for modeling member joints to solve a wider range of problems.

To this end, this study proposes a procedure that uses the finite element method with a transformation technique to analyze structures with semi-rigid connections and constraints. The procedure constructs the element formulations implicitly in matrix form, which is concise and can be expanded in explicit form as needed. Using this approach, any number of semi-rigid connections with displacement constraints at any arbitrary member end can be included without requiring the introduction of additional nodes or degrees of freedom.

2. RESEARCH SIGNIFICANCE

The proposed approach provides a streamlined and universal method for modelling connections between structural elements and constraints, eliminating the necessity for cumbersome matrices typically associated with such problems. Importantly, the condensation process ensures that the size of matrices remains unchanged for both individual elements and the assembled structure, resulting in minimal additional computational time.

The results obtained from linear and nonlinear analyses performed on flexibly jointed frames with displacement constraints demonstrate a remarkable alignment with existing benchmark tests. These findings emphasize the substantial influence of connections and constraints on the distribution of internal forces and deflections within a structure. As a result, the method enhances the efficiency of frame structure design.

3. THE METHOD

3.1 Mathematical Method

Our discussion initially focuses on linear frame problems. Fig. 1 illustrates a three-dimensional frame element with semi-rigid connections and rigid zones at its ends that is subjected to external loading.

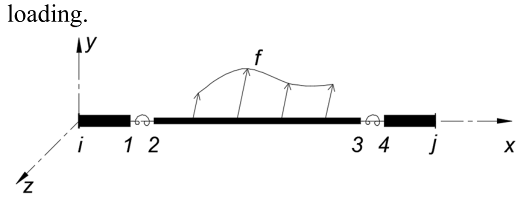


Fig.1 Typical frame element

In this context, the main body is represented by particle *2-3*, while particles *1-2* and *3-4* represent semi-rigid connections, and particles *i-1* and *4-j* represent the rigid zones or displacement constraints of the member. The finite element formulation of the element can be expressed in a familiar matrix form:

$$Kd = F \tag{1}$$

with a system of constraint equations

$$C\tilde{d} = 0 \tag{2}$$

where K is the stiffness matrix, d and $\tilde{d}$ are the nodal displacement vectors, F is the nodal force vector, and C is the displacement constraint matrix of the problem. Vector d comprises all the degrees of freedom of internal points,

$$d = \{d_1 \quad d_2 \quad d_3 \quad d_4\}^T \tag{3}$$

Vector $\tilde{d}$, on the other hand, contains degrees of freedom at end points i and j, and internal points 1 and 4

$$\tilde{d} = \{d_i \quad d_1 \quad d_4 \quad d_j\}^T \tag{4}$$

We aim to obtain a condensed stiffness matrix $\hat{K}$ and force vector $\hat{F}$ that solely involve the degrees of freedom of the endpoints:

$$\hat{K}\hat{d} = \hat{F} \tag{5}$$

where

$$\hat{d} = \{d_i \quad d_j\}^T \tag{6}$$

A transformation technique is now applied to construct the condensed stiffness matrix and condensed force vector for our problem.

3.2 Mathematical Model

We first consider the problem of semi-rigid connections as shown in Fig. 2.

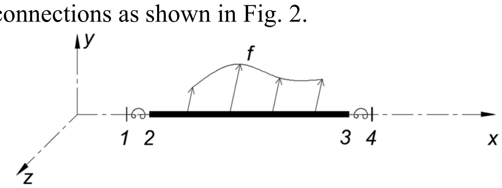


Fig.2 Typical frame element with semi-rigid connections

Referring to Eq. (1), which presents the finite element formulation for our problem, we observe that matrices K and vector F are created using the stiffness matrices and force vectors of both the semi-rigid connections and the body. By employing the transformation method as outlined in [20], we rearrange the formulation into the following format:

$$\begin{bmatrix} K_{rr} & K_{rc} \\ K_{cr} & K_{cc} \end{bmatrix} \begin{Bmatrix} d_r \\ d_c \end{Bmatrix} = \begin{Bmatrix} F_r \\ F_c \end{Bmatrix} \tag{7}$$

where vector d_r contains degrees of freedom that will be retained, and vector d_c contains degrees of freedom that will be condensed

$$d_r = \{d_1 \quad d_4\}^T \tag{8}$$

$$d_c = \{d_2 \quad d_3\}^T \tag{9}$$

Equation (7) is re-written as

$$K_{rr}d_r + K_{rc}d_c = F_r \tag{10}$$

$$K_{cr}d_r + K_{cc}d_c = F_c \tag{11}$$

From Eq. (11), we have

$$d_c = K_{cc}^{-1}\left(F_c - K_{cr}d_r\right) \tag{12}$$

Substituting d_c from Eq. (12) into Eq. (10), we obtain

$$\left(K_{rr} - K_{rc}K_{cc}^{-1}K_{cr}\right)d_r = F_r - K_{rc}K_{cc}^{-1}F_c \tag{13}$$

or

$$\tilde{K}d_r = \tilde{F} \tag{14}$$

where

$$\tilde{K} = K_{rr} - K_{rc}K_{cc}^{-1}K_{cr} \tag{15}$$

$$\tilde{F} = F_r - K_{rc}K_{cc}^{-1}F_c \tag{16}$$

Equation (15) and equation (16) show the condensed stiffness matrix and the condensed force vector, respectively, for our semi-rigid connection problem.

3.3 Condensation for constraints

We now consider the problem of displacement constraints as shown in Fig. 3

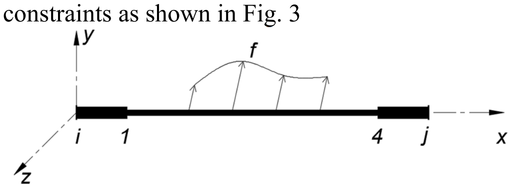


Fig.3 Typical frame member with displacement constraints

The finite element formulation of the problem was shown in Eq. (14) and with constraint in Eq. (2). The constraint equations can be rewritten as

$$C\tilde{d} = \left[\hat{C} \quad C_r\right]\begin{Bmatrix} \hat{d} \\ d_r \end{Bmatrix} = 0 \tag{17}$$

where $\hat{C}$ involves degrees of freedom at end points, whereas C_r involves components at nodes 1 and 4. From Eq. (17), we get

$$\hat{C}\hat{d} + C_r d_r = 0 \tag{18}$$

$$d_r = -C_r^{-1}\hat{C}\hat{d} = T\hat{d} \tag{19}$$

where

$$T = -C_r^{-1}\hat{C} \tag{20}$$

is a transformation matrix. Substituting Eq. (19) for Eq. (14), pre-multiplied by $\mathbf{T}^T$, we get

$$T^T\tilde{K}T\hat{d} = T^T\tilde{F} \tag{21}$$

We recall Eq. (15) and Eq. (16), and denote

$$\hat{K} = T^T\tilde{K}T = T^T(\bar{K}_{rr} - \bar{K}_{rc}\bar{K}_{cc}^{-1}\bar{K}_{cr})T \tag{22}$$

is the condensed stiffness matrix, and

$$\hat{F} = T^T\tilde{F} = T^T(F_r - K_{rc}K_{cc}^{-1}F_c) \tag{23}$$

is the condensed force vector. Eq. (5) and Eqs. (21-23) are now identical.

3.4 Stiffness Matrix and Force Vector of a Typical Beam Element

We utilize the proposed technique to generate the condensed stiffness matrix and condensed force vector for a representative bent beam element with rotationally semi-rigid connections that is subjected to a uniformly distributed force, as illustrated in Fig. 4. The analysis assumes that the relationships between the end moments and the relative rotations of the semi-rigid connections are linear, with stiffness values of k_1 and k_2.

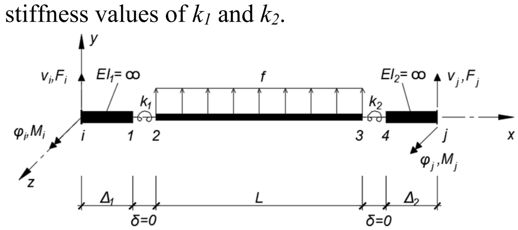


Fig.4 Typical beam bending problem

We first implement for the semi-rigid connection and then for the rigid zone ends, with a procedure of two steps.

Step 1: Condensation for semi-rigid connections

We consider the bar and the semi-rigid connections at its ends. The displacement vector for internal nodes is written as

$$d = \left[v_1 \quad \varphi_1 \quad v_2 \quad \varphi_2 \quad v_3 \quad \varphi_3 \quad v_4 \quad \varphi_4\right]^T \tag{24}$$

The stiffness matrices for the connections at both ends are in the form:

$$k_1 = \begin{array}{c}\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}\end{array}\begin{array}{l}(\varphi_1)\\(\varphi_2)\end{array} \tag{25}$$

$$k_2 = \begin{array}{c}\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\end{array}\begin{array}{l}(\varphi_3)\\(\varphi_4)\end{array} \tag{26}$$

and the stiffness matrix is:

$$k_b = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}\begin{array}{l}(v_2)\\(\varphi_2)\\(v_3)\\(\varphi_3)\end{array} \tag{27}$$

and the equivalent force vector of the body is:

$$F_b = \begin{Bmatrix} \frac{fL}{2} \\ \frac{fL^2}{12} \\ \frac{fL}{2} \\ \frac{-fL^2}{12} \end{Bmatrix}\begin{array}{l}(F_2)\\(M_2)\\(F_3)\\(M_3)\end{array} \tag{28}$$

Note that $v_1=v_2$ and $v_3=v_4$. After assembling the stiffness matrices and the force vectors, we have the finite formulation for the system:

$$\begin{bmatrix} \frac{12EI}{L^3} & 0 & -\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} & \frac{6EI}{L^2} \\ 0 & k_1 & 0 & 0 & -k_1 & 0 \\ -\frac{12EI}{L^3} & 0 & \frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} & -\frac{6EI}{L^2} \\ 0 & 0 & 0 & k_2 & 0 & -k_2 \\ \frac{6EI}{L^2} & -k_1 & -\frac{6EI}{L^2} & 0 & \frac{4EI}{L}+k_1 & \frac{2EI}{L} \\ \frac{6EI}{L^2} & 0 & -\frac{6EI}{L^2} & -k_2 & \frac{2EI}{L} & \frac{4EI}{L}+k_2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \varphi_1 \\ v_4 \\ \varphi_4 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix} = \begin{Bmatrix} F_1+\frac{fL}{2} \\ M_1 \\ F_4+\frac{fL}{2} \\ M_4 \\ \frac{fL^2}{12} \\ -\frac{fL^2}{12} \end{Bmatrix} \tag{29}$$

We denote:

$$d_r = \{v_1 \quad \varphi_1 \quad v_4 \quad \varphi_4\}^T \tag{30}$$

$$d_c = \{\varphi_2 \quad \varphi_3\}^T \tag{31}$$

$$K_{rr} = \begin{bmatrix} \frac{12EI}{L^3} & 0 & -\frac{12EI}{L^3} & 0 \\ 0 & k_1 & 0 & 0 \\ -\frac{12EI}{L^3} & 0 & \frac{12EI}{L^3} & 0 \\ 0 & 0 & 0 & k_2 \end{bmatrix} \tag{32}$$

$$K_{rc} = \begin{bmatrix} \frac{6EI}{L^2} & \frac{6EI}{L^2} \\ -k_1 & 0 \\ -\frac{6EI}{L^2} & -\frac{6EI}{L^2} \\ 0 & -k_2 \end{bmatrix} \tag{33}$$

$$K_{cr} = \begin{bmatrix} \frac{6EI}{L^2} & -k_1 & -\frac{6EI}{L^2} & 0 \\ \frac{6EI}{L^2} & 0 & -\frac{6EI}{L^2} & -k_2 \end{bmatrix} \tag{34}$$

$$K_{cc} = \begin{bmatrix} \frac{4EI}{L}+k_1 & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L}+k_2 \end{bmatrix} \tag{35}$$

$$F_r = \left\{(F_1+\frac{fL}{2}) \quad M_1 \quad (F_4+\frac{fL}{2}) \quad M_4\right\}^T \tag{36}$$

and

$$F_c = \left\{\frac{fL^2}{12} \quad \frac{-fL^2}{12}\right\}^T \tag{37}$$

Equation (29) and equation (7) are now identical. Implementing Eq. (15) and Eq. (16), we get

$$\tilde{K} = \tilde{K}_{\alpha\beta} = \begin{bmatrix} \frac{12EI}{L^3} & 0 & -\frac{12EI}{L^3} & 0 \\ 0 & k_1 & 0 & 0 \\ -\frac{12EI}{L^3} & 0 & \frac{12EI}{L^3} & 0 \\ 0 & 0 & 0 & k_2 \end{bmatrix} \tag{38}$$

$$-\begin{bmatrix} \frac{6EI}{L^2} & \frac{6EI}{L^2} \\ -k_1 & 0 \\ -\frac{6EI}{L^2} & -\frac{6EI}{L^2} \\ 0 & -k_2 \end{bmatrix}\begin{bmatrix} \frac{4EI}{L}+k_1 & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L}+k_2 \end{bmatrix}^{-1}\begin{bmatrix} \frac{6EI}{L^2} & -k_1 & -\frac{6EI}{L^2} & 0 \\ \frac{6EI}{L^2} & 0 & -\frac{6EI}{L^2} & -k_2 \end{bmatrix}$$

$$\tilde{F}=\tilde{F}_{\alpha}=\begin{Bmatrix} F_1+\frac{fL}{2} \\ M_1 \\ F_4+\frac{fL}{2} \\ M_4 \end{Bmatrix} - \begin{bmatrix} \frac{6EI}{L^2} & \frac{6EI}{L^2} \\ -k_1 & 0 \\ -\frac{6EI}{L^2} & -\frac{6EI}{L^2} \\ 0 & -k_2 \end{bmatrix} \begin{bmatrix} \frac{4EI}{L}+k_1 & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L}+k_2 \end{bmatrix}^{-1} \begin{Bmatrix} \frac{fL^2}{12} \\ -\frac{fL^2}{12} \end{Bmatrix} \tag{39}$$

which are matrix forms for the condensed stiffness matrix and the condensed force vector for the bar with the semi-rigid connections. Equation (39) indicates that the equivalent force vector can be expressed in the following manner:

$$\tilde{F}^{eqv}=F_{\alpha}^{eqv}=\begin{Bmatrix} \frac{fL}{2} \\ 0 \\ \frac{fL}{2} \\ 0 \end{Bmatrix} - \begin{bmatrix} \frac{6EI}{L^2} & \frac{6EI}{L^2} \\ -k_1 & 0 \\ -\frac{6EI}{L^2} & -\frac{6EI}{L^2} \\ 0 & -k_2 \end{bmatrix} \begin{bmatrix} \frac{4EI}{L}+k_1 & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L}+k_2 \end{bmatrix}^{-1} \begin{Bmatrix} \frac{fL^2}{12} \\ -\frac{fL^2}{12} \end{Bmatrix}. \tag{40}$$

Components of the stiffness matrix and the force vector for the bar with the semi-rigid connections are shown in Table 1. It is noted that the stiffness matrix is symmetric.

Step 2: Condensation for rigid-zone ends

We now consider the problem of constraints with the following displacement vector

$$\tilde{d}=\{v_i \quad \varphi_i \quad v_i \quad \varphi_j \quad v_1 \quad \varphi_2 \quad v_3 \quad \varphi_4\}^T=\{\hat{d} \quad d_r\}^T \tag{41}$$

where

$$\hat{d}=\{v_i \quad \varphi_i \quad v_i \quad \varphi_j\}^T \tag{42}$$

$$d_r=\{v_1 \quad \varphi_2 \quad v_3 \quad \varphi_4\}^T \tag{43}$$

We have displacement constraint conditions $\varphi_i=\varphi_1$, $\varphi_j=\varphi_4$, $u_1=u_i+\Delta_1\varphi_i$, $u_j=u_4+\Delta_2\varphi_j$. We can write the constraints in the following matrix form:

.

$$\begin{bmatrix} -1 & -\Delta_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & \Delta_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_i \\ \varphi_i \\ v_j \\ \varphi_j \\ v_1 \\ \varphi_1 \\ v_4 \\ \varphi_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{44}$$

We denote:

$$\hat{C}=\begin{bmatrix} -1 & -\Delta_1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & \Delta_2 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{45}$$

and

$$C_r=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{46}$$

Table 1 The stiffness matrix and the force vector for the bar with the semi-rigid connections

$i=EI/L$

$k=12i^2+4k_2i+4k_1i+k_1k_2$

Terms	Expressions
$\tilde{K}_{11}$	$\frac{12EI}{L^3}\frac{(k_1k_2+k_2i+k_1i)}{k}$
$\tilde{K}_{12}$	$\frac{6EI}{L^2}\frac{k_1(k_2+2i)}{k}$
$\tilde{K}_{13}$	$-\frac{12EI}{L^3}\frac{(k_1k_2+k_2i+k_1i)}{k}$
$\tilde{K}_{14}$	$\frac{6EI}{L^2}\frac{k_2(k_1+2i)}{k}$
$\tilde{K}_{22}$	$\frac{4EI}{L}\frac{k_1(3i+k_2)}{k}$
$\tilde{K}_{23}$	$-\frac{6EI}{L^2}\frac{k_1(k_2+2i)}{k}$
$\tilde{K}_{24}$	$\frac{2EI}{L}\frac{k_1k_2}{k}$
$\tilde{K}_{33}$	$\frac{12EI}{L^3}\frac{(k_1k_2+k_2i+k_1i)}{k}$
$\tilde{K}_{34}$	$-\frac{6EI}{L^2}\frac{k_2(k_1+2i)}{k}$
$\tilde{K}_{44}$	$\frac{4EI}{L}\frac{k_2(3i+k_1)}{k}$
$\tilde{F}_1$	$\frac{qL}{2}\frac{(12i^2+5k_1i+3k_2i+k_1k_2)}{k}$
$\tilde{F}_2$	$\frac{qL^2}{12}\frac{k_1(6i+k_2)}{k}$
$\tilde{F}_3$	$\frac{qL}{2}\frac{(12i^2+3k_1i+5k_2i+k_1k_2)}{k}$
$\tilde{F}_4$	$-\frac{qL^2}{12}\frac{k_2(6i+k_1)}{k}$

The transformation is in the form:

$$T=-C_r^{-1}\hat{C}=\begin{bmatrix} 1 & \Delta_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\Delta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{47}$$

Recalling Eq. (22) and Eq. (23), we get the condensed stiffness matrix

$$\hat{K} = T^T \tilde{K} T = \hat{K}_{\alpha\beta} \tag{48}$$

and the condensed force vector

$$\hat{F} = T^T \tilde{F} = \hat{F}_{\alpha} \tag{49}$$

Here, indices α and β are in a range from 1 to 4 for our beam bending problem. Components of the stiffness matrix and the condensed force vector are presented in Table 2

Table 2 The stiffness matrix and the force vector for the bar with the semi-rigid connections and the rigid zone ends

$i = EI / L$

$k = 12i^2 + 4k_2 i + 4k_1 i + k_1 k_2$

Terms	Expressions
$\hat{K}_{11}$	$\frac{12EI}{L^3}\frac{1}{k}\left(k_1k_2 + k_2 i + k_1 i\right)$
$\hat{K}_{12}$	$\frac{6EI}{L^2}\frac{1}{k}\left[k_1\left(k_2 + 2i\right) + \left(k_1k_2 + k_2 i + k_1 i\right)\frac{2\Delta_1}{L}\right]$
$\hat{K}_{13}$	$-\frac{12EI}{L^3}\frac{1}{k}\left(k_1k_2 + k_2 i + k_1 i\right)$
$\hat{K}_{14}$	$\frac{6EI}{L^2}\frac{1}{k}\left[k_2\left(k_1 + 2i\right) + \left(k_1k_2 + k_2 i + k_1 i\right)\frac{2\Delta_2}{L}\right]$
$\hat{K}_{22}$	$\frac{4EI}{L}\frac{1}{k}\left[k_1\left(3i + k_2\right) + k_1\left(k_2 + 2i\right)\frac{3\Delta_1}{L} + \left(k_1k_2 + k_2 i + k_1 i\right)\frac{3\Delta_1^2}{L^2}\right]$
$\hat{K}_{23}$	$-\frac{6EI}{L^2}\frac{1}{k}\left[k_1\left(k_2 + 2i\right) + \left(k_1k_2 + k_2 i + k_1 i\right)\frac{2\Delta_1}{L}\right]$
$\hat{K}_{24}$	$\frac{2EI}{L}\frac{1}{k}\left[k_1k_2 + k_2\left(k_1 + 2i\right)\frac{3\Delta_1}{L} + k_1\left(k_2 + 2i\right)\frac{3\Delta_2}{L} + \left(k_1k_2 + k_2 i + k_1 i\right)\frac{6\Delta_1\Delta_2}{L^2}\right]$
$\hat{K}_{33}$	$\frac{12EI}{L^3}\frac{1}{k}\left(k_1k_2 + k_2 i + k_1 i\right)$
$\hat{K}_{34}$	$-\frac{6EI}{L^2}\frac{1}{k}\left[k_2\left(k_1 + 2i\right) + \left(k_1k_2 + k_2 i + k_1 i\right)\frac{2\Delta_2}{L}\right]$
$\hat{K}_{44}$	$\frac{4EI}{L}\frac{1}{k}\left[k_2\left(3i + k_1\right) + k_2\left(k_1 + 2i\right)\frac{3\Delta_2}{L} + \left(k_1k_2 + k_2 i + k_1 i\right)\frac{3\Delta_2^2}{L^2}\right]$
$\hat{F}_1$	$\frac{qL}{2}\frac{1}{k}\left(12i^2 + 5k_1 i + 3k_2 i + k_1k_2\right)$
$\hat{F}_2$	$\frac{qL^2}{12}\frac{1}{k}\left[k_1\left(6i + k_2\right) + \left(12i^2 + 3k_2 i + 5k_1 i + k_1k_2\right)\frac{6\Delta_1}{L}\right]$
$\hat{F}_3$	$\frac{qL}{2}\frac{1}{k}\left(12i^2 + 3k_1 i + 5k_2 i + k_1k_2\right)$
$\hat{F}_4$	$-\frac{qL^2}{12}\frac{1}{k}\left[k_2\left(6i + k_1\right) + \left(12i^2 + 3k_2 i + 5k_1 i + k_1k_2\right)\frac{6\Delta_2}{L}\right]$

3.5 Nonlinear Analysis

We will now examine a frame problem with nonlinear semi-rigid connections, and apply the standard procedure outlined below:

1. At the current loading step and at the incremental step i, determine the displacement vector $\hat{d}_i$, stiffness matrix $\hat{K}_i$, external force vector $\hat{F}_i$, and the structure resisting force $\hat{R}_i$. Solving the global structure equilibrium equation, we receive the incremental displacement $\Delta\hat{d}_i$

$$\Delta\hat{d}_i = \hat{K}_i^{-1}(\hat{F}_i - \hat{R}_i) \tag{50}$$

and the displacement for the next step

$$\hat{d}_{i+1} = \hat{d}_i + \Delta\hat{d}_i \tag{51}$$

2. Compute member strains and forces based on the updated displacement vector $\hat{d}_{i+1}$.

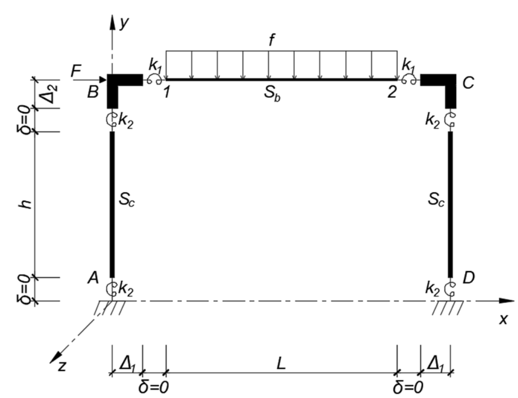
3. Update the element stiffness matrix $\hat{K}_{i+1}$, the force vector $\hat{F}_{i+1}$, the structure resisting force vector $\hat{R}_{i+1}$, and the unbalanced force vector $\hat{F}_{i+1} - \hat{R}_{i+1}$.

4. Verify if the structure has converged. If the unbalanced forces satisfy the specified tolerance, indicating convergence, move on to the next incremental loading step. If not, return to step 1 for the subsequent iteration to eliminate the unbalanced forces of the structure.

4. EXAMPLES

4.1 Example 1. Frame With Rotationally Semi-rigid Connections and Constraints

Let us consider a planar frame problem with rotationally semi-rigid connections and rigid-zone ends, subjected to the uniformly distributed force shown in Fig. 5. It is required to compute nodal displacements and construct internal force diagrams.



$L=10$ m; S_b $(b_b \times h_b)$: 0.4 m$\times$ 0.8 m; $\Delta_1=0.4$ m

$h=6$ m; S_c $(b_c \times h_c)$: 0.4 m$\times$ 0.6 m; $\Delta_2=0.3$ m

$k_1=50000$ kN/m; $F=50$ kN; $E=2\times10^7$ kN/m^2

$k_2=20000$ kN/m; $f=10$ kN/m

Fig.5 A planar frame problem

The results of nodal displacements and internal forces using Etabs and the condensed method are shown in Table 3. It should be noted that a mesh with 24 degrees of freedom was used for the Etabs model, whereas 6 degrees of freedom are needed for the analysis using our condensed method.

The results using the proposed formulations show good agreement with the finite element analysis from ETABS software program.

Table 3 Nodal displacements and internal forces for Example 2

Quantity	ETABS	Condensed method	Difference (%)
Nodal disp.			
u_B [m]	0.030816	0.030704	0.363
v_B [m]	-0.000046	-0.000046	0.000
φ_B [rad]	0.002727	0.002720	0.257
u_C [m]	0.030771	0.030659	0.364
v_C [m]	-0.000079	-0.000079	0.000
φ_C [rad]	0.000348	0.000344	1.149
Int. forces			
M_A [kNm]	82.88	82.79	0.109
M_B [kNm]	50.73	50.75	0.039
M_C [kNm]	-92.54	-92.62	0.086
M_D [kNm]	88.85	88.84	0.011
Q_{AB} [kN]	21.21	21.20	0.047
Q_{BC} [kN]	-36.73	-36.73	0.000
Q_{CB} [kN]	63.27	63.28	0.016
Q_{DC} [kN]	28.79	28.80	0.035
N_{AB} [kN]	-36.73	-36.73	0.000
N_{BC} [kN]	-28.79	-28.80	0.035
N_{DC} [kN]	-63.27	-63.28	0.016

4.2 Example 2. Stelmack two-storey frame

A two-story and one-bay frame shown in Fig.6, presented in the work by Stelmack et al. [11], is adopted as a benchmark test to study the nonlinear response of the structure. The frame members are fabricated from A36 W5x16 sections. Column bases are pinned supports. Connections used in the frame were bolted top and seat angle connections of A36 L4x4x1/2. The experimental moment-rotation relationship is shown as the dashed line in Fig.7. A gravity loading of P=10.7kN (2.4 kips) was first applied, and then a lateral load was applied. The lateral load-displacement relationship shown as the dashed line in Fig.8 was provided by the experimental work.

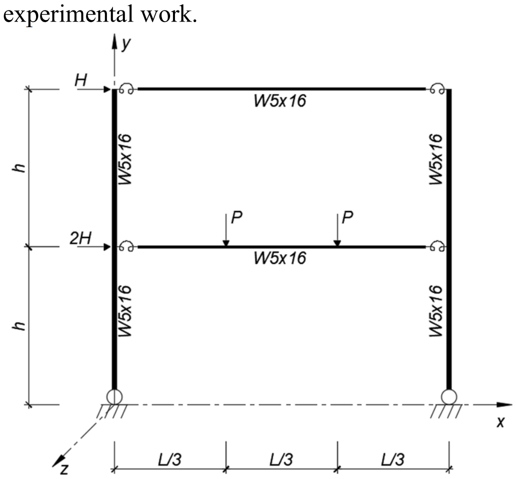


Fig.6 A two-story and one-bay frame

The three-parameter power model proposed by Kishi and Chen is used as the material model of the connection. Here, we utilize the parameters used by Thai and Kim [18], with the initial elastic stiffness R_{ki}=*4519.4 kNm/rad (40000 kip.in/rad),* ultimate moment M_u=*24.9 kNm (220 kip.in)* and shape parameter n=*0.91*. The current moment is determined by the following expression:

$$M = \frac{K_i \theta}{\left[1+\left(\frac{\theta}{\theta_0}\right)^n\right]^{1/n}} \tag{52}$$

the current tangent stiffness is

$$K = \frac{dM}{d\theta} = \frac{K_i}{\left[1+\left(\frac{\theta}{\theta_0}\right)^n\right]^{(n+1)/n}} \tag{53}$$

and the relative rotation is

$$\theta = \frac{M}{K_i\left[1-\left(\frac{M}{M_u}\right)^n\right]^{1/n}} \tag{54}$$

The incremental relative rotation for updating each step is

$$\Delta\theta_{i+1} = \frac{\Delta M_i}{k_i} \tag{55}$$

then, the updated relative rotation is given by

$$\theta_{i+1} = \theta_i + \Delta\theta_i \tag{56}$$

and the internal force vector for updating

$$R = \begin{Bmatrix} \frac{M_1+M_4}{L} - P(1-\frac{b}{L}) \\ M_1 \\ \frac{M_1+M_4}{L} + P\frac{b}{L} \\ M_4 \end{Bmatrix} \tag{57}$$

Shown in Fig. 7 is the moment-rotation relationship for our problem.

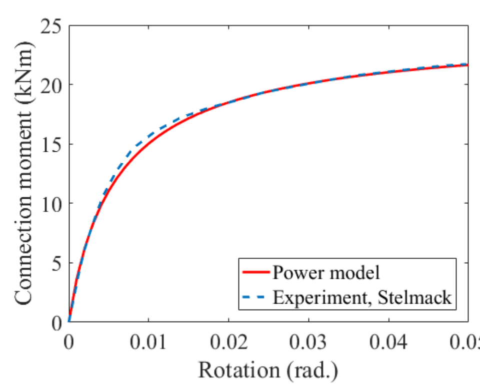


Fig.7 Moment-rotation curves

Applying the presented iteration procedure, the nonlinear problem was analyzed. The lateral load-displacement relationship at the first floor is shown in Fig. 8

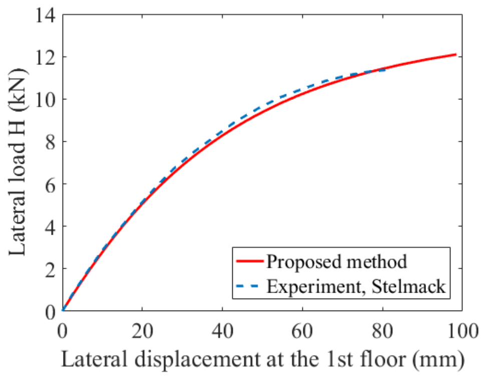


Fig.8 Comparison of lateral load-displacement curves by experiment and proposed method for verification study

The result shows a good agreement with the experimental tests presented by Stelmack et al. [11]

5. CONCLUSIONS

Using the transformation technique, a numerical procedure and condensed finite element formulations were developed to analyze steel frames with semi-rigid connections and displacement constraints. The proposed method was found to be general and simple, enabling more realistic modeling of connections between structural elements without complex mathematics. Additionally, the stiffness matrix for the element and assembled structure does not increase after the condensation process, minimizing extra computational time. The proposed method is illustrated through examples of linear and nonlinear analyses of flexibly jointed frames, exhibiting strong agreement with benchmark tests that are currently available. These results highlight the significant effect that connections and constraints have on internal force and deflection distribution in a structure, leading to greater efficiency in the design of frame structures. Practical methods for determining connection parameters are provided for specific connection configurations.

In addition, the new method is highly practical for engineering applications as it can be conveniently implemented computationally and calculated by hand with ease. Moreover, the procedure can be extended to address various structural problems, such as distributed semi-rigid connections, second-order effects, or cyclic dynamic loading.

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