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VARIATIONAL ITERATION METHOD AND ANALYTIC SOLUTION FOR LAPLACE EQUATION FOR STEADY GROUNDWATER FLOW

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ABSTRACT: In this paper, we present a new approach for solving the Laplace equation for steady groundwater flow using the Variational Iteration Method (VIM) and Analytic Solution. The Laplace equation is a fundamental equation that describes the behavior of groundwater flow in porous media. However, the analytical solution for this equation is not always possible, especially for complex geometries and boundary conditions. Finding a solution to the Laplace equation for steady groundwater flow in a given domain with certain boundary conditions is the stated problem. The strategy utilized in this study comprises using the VIM to solve the Laplace equation in series. A variety of differential equations can be solved using the VIM, which is a strong and effective method. In this study, we present the results of our analysis for different boundary conditions and geometries. The results show that the VIM is an effective method for solving the Laplace equation for steady groundwater flow. The solutions obtained with the VIM are compared with the analytic solutions, and good agreement is observed. In conclusion, the VIM and Analytic Solution approach is a promising method for solving the Laplace equation for steady groundwater flow. The results obtained with this method can be used to design and optimize groundwater remediation systems and to study the behavior of groundwater flow in complex geometries.

Keywords: Variational Iteration Method (VIM), Analytic Solution, Laplace equation, Groundwater flow.

1. INTRODUCTION

A noteworthy proportion of the precipitation that occurs as rain on terrestrial surfaces traverses unsaturated soil during the subsequent processes of infiltration, drainage, evaporation, and absorption of soil water by plant roots. Despite this, soil physics, which is primarily interested in agronomic or ecological aspects of hydrology, has done the majority of study on this subject. Hydrologists, on the other hand, have tended to pay relatively little attention to the phenomenon of water movement in unsaturated soils.

The unsteady and unsaturated flow of water through soils is due to content changes as a function of time, and entire pore spaces are not completely filled with flowing liquid respectively. Knowledge concerning such flows is relevant to various workers including hydrologists, agriculturalists, and many fields of science and engineering. The water infiltration system and the underground disposal of seepage and waste water are encountered by these flows, which are described by nonlinear partial differential equation.

The mathematical model conforms to the hydrological situation of one-dimensional vertical groundwater recharge by Spreading [1]. Such flows are of great importance in water resources science, soil engineering, and agricultural sciences.

Numerous researchers have discussed this phenomenon from various perspectives. For example, Klute [2] and Hank Bower [3] employ a finite difference method; Philips [4] uses a transformation of variable technique; Mehta [5] discussed a multiple-scale method; Verma [1,6] has obtained Laplace transformation and similarity solution, and Sharma [7] discusses a variational approach. Bruce and Klute [8]; Gardner and Mayhugh [9]; Nielson and Bigger [10]; Rawlins and Gardner [11], Terwilliger [12], Van Vorts [13], and Rahme [14] have described the phenomenon of gravity drainage of liquids through porous media and supported their theoretical investigation by experimental results.

In the present research paper, a ground water recharge problem with parabolic permeability is solved by Variational Iteration Method. He (1999, 2000, 2006) developed the variational iteration method for solving linear, nonlinear, and boundary value problems.

The method was first considered by Inokuti, Sekine, and Mura (1978) and fully explored by He. J. H. In this method, the solution is given in an infinite series usually converging to an accurate solution. Olayiwolaetal (2009) used modified power series method for the solution of systems of differential equations. It is observed that the method solves effectively, easily, and accurately a class of linear, nonlinear, ordinary differential equations with approximate solution, which converge very rapidly to accurate solution. Recently introduced variational iteration method by He [16,17-19], which gives rapidly convergent successive approximations of the exact solution if such a solution exists, has proved successful in deriving analytical solutions of linear and nonlinear differential equations.

This method is preferable over numerical methods as it is free from rounding off errors and neither requires large computer power/memory. He [16, 18] has applied this method for obtaining analytical solutions of autonomous ordinary differential equation, nonlinear partial differential equations with variable coefficients, and integro-differential equations. Mohsenin [20] discusses the thermal properties of foods and agricultural materials. Sun [21] highlights the use of computational fluid dynamics (CFD) as a design and analysis tool in the agri-food industry. Jun and Sastry [22] presents a model for the optimization of ohmic heating of foods inside a flexible package. Roy et al. [23] presents a CFD-based approach for determining temperature and humidity at leaf surfaces. Ortega et al. [24] presents a benchmark experiment for conjugate forced convection from a discrete heat source on a plane conducting surface. Sablani et al. [25] presents dimensionless correlations for convective heat transfer to liquid and particles in cans subjected to end-over-end rotation. Jensen and Friis [26] presents a prediction of flow in a mixproof valve using CFD, validated by LDA. De-bonis and Ruocco [27] presents a generalized conjugate model for forced convection drying based on an evaporative kinetics. Davalath and Bayazitoglu [28] discusses forced convection cooling across rectangular blocks. Garron and Garimella [29] presents composite correlations for convective heat transfer from arrays of three-dimensional obstacles. Young and Vafai [30] presents a study on convective flow and heat transfer in a channel containing multiple heated obstacles. Wazwaz [31-33]present the variational iteration method for solving linear and nonlinear wave equations and Volterra integrodifferential forms of the Lane-Emden and the Emden-Fowler problems with initial and boundary value conditions. Maturi [34-42] present various numerical methods for solving integral equations, heat conduction equations, and integro-differential equations using tools such as Maple and finite difference method.

2. RESEARCH SIGNIFICANCE

The Laplace equation is a fundamental partial differential equation used to model various physical phenomena, including steady-state groundwater flow. While analytical solutions to this equation are challenging to obtain, the Variational Iteration Method (VIM) is a powerful analytical and numerical technique that can efficiently and accurately solve partial differential equations, even those that are nonlinear and singular. By employing VIM to solve the Laplace equation for steady-state groundwater flow, it is possible to predict groundwater behavior and develop effective strategies for managing and conserving this vital resource. The accuracy and computational efficiency of VIM make it particularly useful for large-scale problems, and its adoption can have significant implications for environmental management and resource conservation.

3. VARIATIONAL ITERATION METHOD

For the differential equation

$$Lu + Nu = g(x,t) \tag{1}$$

The correction functional for equation (1) can be expressed as follows: the variational iteration approach allows the usage of this functional where L and N are linear and nonlinear operators, respectively, and g(x,t) is the source inhomogeneous term.

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\,(Lu_n(\xi) + N\widetilde{u_n}(\xi) - g(\xi)))d\xi,\ \text{n} \geq 0 \tag{2}$$

First, we need to find the Lagrange multiplier λ(ξ). Using the resulting Lagrange multiplier and any selective function u_0, one can easily construct the successive approximations $u_{n+1}(x,t)$, n ≥ of the solution $u(x,t)$.

The zeroth approximation, u_0, should be chosen using the initial values $u(x,0)$ and $u_t(x,0)$. It is possible to get the precise answer by using

$$u = \lim_{n\to\infty} u_n \tag{3}$$

It is worth noting

$$\int \lambda(\xi) u_n''(\xi)\, d\xi = \lambda(\xi) u_n'(\xi) - \lambda'^{(\xi) u_n(\xi)} + \int \lambda'' u_n(\xi)\, d\xi \tag{4}$$

4. GROUNDWATER FLOW IN AVLLEY

More than a hundred years ago, Henri-Philibert-Gaspard Darcy, a French hydraulic engineer, conducted a laboratory experiment on water flow through sand and published his findings. He demonstrated that the apparent fluid velocity q in relation to sand grains is directly proportional to the gradient of hydraulic potential $-k\nabla\phi$.

The hydraulic potential ϕ represents the sum of the point of measurement's elevation and the pressure potential $(p/\rho g)$. In the event of constant flow, Darcy's law combined with mass conservation $\nabla \cdot q = 0$ gives rise to Laplace's equation ∇2ϕ = 0, assuming that the aquifer is isotropic (same in all directions) and homogeneous.

To demonstrate how separation of variables can aid in solving Laplace's equation, we will determine the hydraulic potential within a small drainage basin situated in a shallow valley, as presented in Figure 1.

Pursuant to Toth the governing equation is the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$
$$0 < x < L, \quad 0 < y < z_0, \tag{5}$$

Along with the boundary conditions

$$\boldsymbol{u(x, z_0) = gz_0 + gcx}, \tag{6}$$

$$u_x(0,y) = u_x(L,y) = 0, \; and \;\; u_y(x,0) = 0, \tag{7}$$

the present study concerns the hydraulic potential denoted as $u(x,y)$, the acceleration due to gravity denoted as g, and the slope of the topography represented by c. The no-flow condition through the bottom and sides of the aquifer is specified by the conditions $u_x(L,y) = 0$ and $u_y(x,0) = 0$. Additionally, symmetry about the $x = 0$ line is ensured by the condition $u_x(0,y) = 0$.

Furthermore, Equation 5 provides the fluid potential at the water table, where the elevation of the water table above the standard datum is denoted as z_0. Finally, the term gcx in unveils significant insights regarding the hydraulic potential. The potential rise from the valley bottom toward the water divide is expressed in equation 6. Generally speaking, it closely mimics the geography.

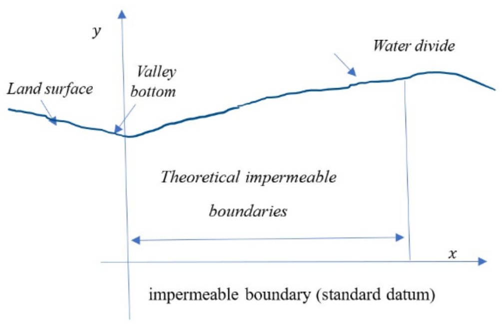

Fig.1 Gross section of a valley.

5. SEVERAL EXAMPLE

Example1. Consider the Laplace equation for steady Groundwater Flow

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x, y < \pi$$
$$u(0,y) = 0, u(\pi,y) = \sinh(\pi)\sin(y)$$
$$u(x,0) = 0, u(x,\pi) = 0$$

Applying Variational Method using Maple

Table 1 Numerical results and exact solution of Laplace equation for steady Groundwater Flow for example 1

$\boldsymbol{x}$	$\boldsymbol{u(x)}$	$\boldsymbol{Exact =}$ $\mathbf{sinh}(\boldsymbol{x})\,\mathbf{sin}(\boldsymbol{y})$	$\boldsymbol{Error}$
0.10000	0.00999999	0.00999999	0.00000000
0.20000	0.03999929	0.03999929	0.00000000
0.30000	0.08999190	0.08999189	0.00000001
0.40000	0.15995449	0.15995436	0.00000013
0.50000	0.24982640	0.24982565	0.00000075
0.60000	0.35948165	0.35947850	0.00000315
0.70000	0.48869304	0.48868244	0.00001060
0.80000	0.63708824	0.63705812	0.00003012
0.90000	0.80409817	0.80402299	0.00007518
1.00000	0.98889771	0.98872841	0.00016930

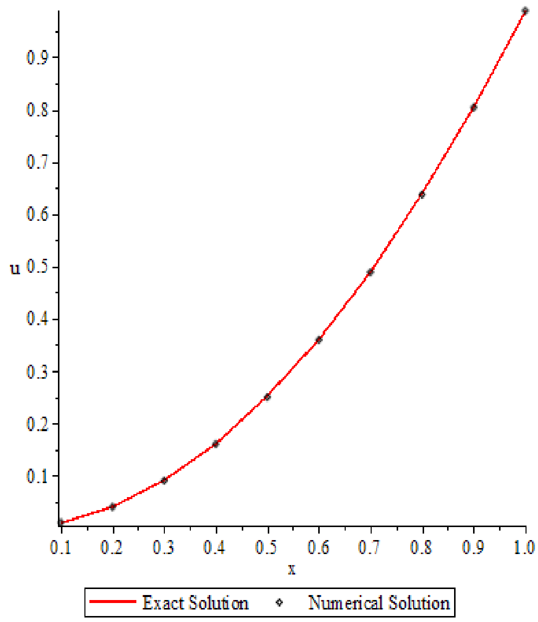

Fig.2 Plot 2D of the exact solutions result of Laplace equation for steady Groundwater Flow for example 1.

Example2. Consider the Laplace equation for steady Groundwater Flow

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x, y < \pi$$
$$u_x(0,y) = 0, u_x(\pi,y) =$$
$$u_y(x,0) = 0, u_y(x,\pi) = \mathrm{coh}(\pi)\cos(y)$$

Applying Variational Method using Maple

Table 2 Numerical results and exact solution of Laplace equation for steady Groundwater

Flow for example 2

x	$u(x)$	$Exact =$ $\cos(x)\sinh(y)$	$Error$
0.10000	0.09966633	0.09966633	0.00000000
0.20000	0.19732269	0.19732268	0.00000000
0.30000	0.29091935	0.29091931	0.00000004
0.40000	0.37832795	0.37832765	0.00000030
0.50000	0.45730415	0.45730279	0.00000137
0.60000	0.52545288	0.52544827	0.00000461
0.70000	0.58019682	0.58018423	0.00001258
0.80000	0.61874940	0.61872015	0.00002925
0.90000	0.63809303	0.63803337	0.00005966
1.00000	0.63496392	0.63485521	0.00010871

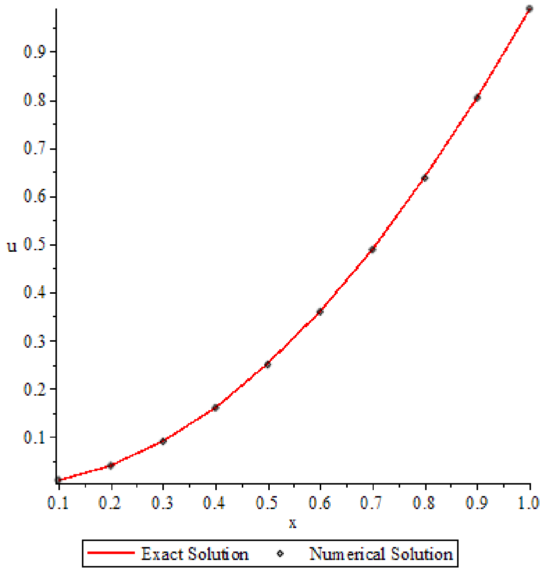

Fig.3 Plot 2D of the exact solutions result of Laplace equation for steady Groundwater Flow for example 2.

Example3. Consider the Laplace equation for steady Groundwater Flow

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x, y < \pi$$
$$u_x(0, y) = 0, u_x(\pi, y) = 0$$
$$u_y(x, 0) = \cos(x), u_y(x, \pi) = \mathrm{coh}(\pi)\cos(x)$$

Example4. Consider the Laplace equation for steady Groundwater Flow

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x, y < \pi$$
$$u_x(0, y) = 0, u_x(\pi, y) = 0$$
$$u(x, 0) = \cos(x), u(x, \pi) = \cosh(\pi)\cos(x)$$

Applying Variational Method using Maple

Table 3 Numerical results and exact solution of Laplace equation for steady Groundwater Flow for example 3

x	$u(x)$	$Exact =$ $\cos(x)\sin h(y$	$Error$
0.10000	0.09966633	0.09966633	0.00000000
0.20000	0.19732269	0.19732268	0.00000000
0.30000	0.29091935	0.29091931	0.00000004
0.40000	0.37832795	0.37832765	0.00000030
0.50000	0.45730415	0.45730279	0.00000137
0.60000	0.52545288	0.52544827	0.00000461
0.70000	0.58019682	0.58018423	0.00001258
0.80000	0.61874940	0.61872015	0.00002925
0.90000	0.63809303	0.63803337	0.00005966
1.00000	0.63496392	0.63485521	0.00010871

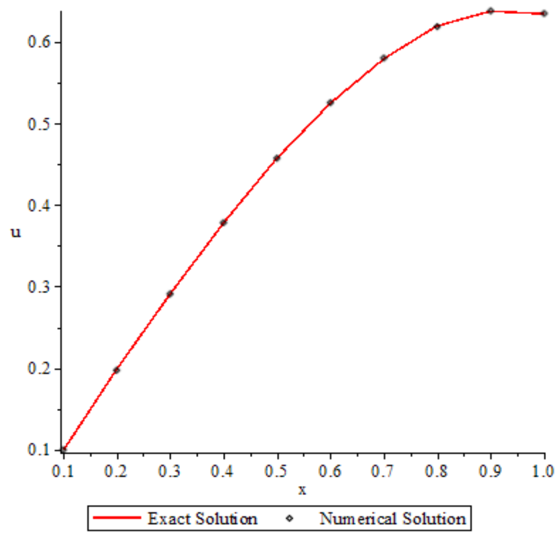

Fig.4 Plot 2D of the exact solutions result of Laplace equation for steady Groundwater Flow for example 3.

Table 4 Numerical results and exact solution of Laplace equation for steady Groundwater Flow for example 4

x	$u(x)$	$Exact =$ $\cos(x)\cos h(y$	$Error$
0.10000	0.99998333	0.99998333	0.00000000
0.20000	0.99973333	0.99973325	0.00000009
0.30000	0.99865003	0.99864906	0.00000097
0.40000	0.99573359	0.99572834	0.00000525
0.50000	0.98958488	0.98956575	0.00001913
0.60000	0.97840666	0.97835284	0.00005383
0.70000	0.96000621	0.95988013	0.00012608
0.80000	0.93179990	0.93154332	0.00025658
0.90000	0.89082078	0.89035527	0.00046551
1.00000	0.83373003	0.83296606	0.00076397

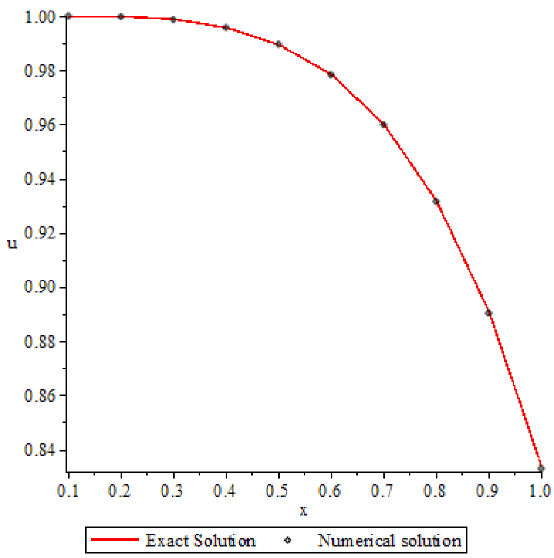

Fig.5 Plot 2D of the exact solutions result of Laplace equation for steady Groundwater Flow for example 4.

6. LAPLACE EQUTION FOR STEADY GROUNDWATER FLOW IN CYLINDRICAL COORDINATES

The concept of electrostatic potential, which is the work required to overcome electric forces in order to transport a unit charge from a point of reference to a designated point, is fundamental in the field of electromagnetism. Moreover, it can be demonstrated that Laplace's equation governs the electrostatic potential in domains devoid of electric charge. Keeping this in mind, we'll try to determine the electrostatic potential, denoted by $u(r,z)$, inside a closed cylinder that has a L and an an in its length and radius. The potential of the upper surface is V while the potential of the base and lateral surfaces is 0. Laplace's equation in cylindrical coordinates can be simplified to its simplest form because the potential depends only on r and z.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial^2 u}{\partial z^2}=0,$$
$$0 \leq r < a, \quad 0 < z < L, \tag{8}$$

depending on the boundary circumstances.

$$u(a,z)=u(r,0)=0, and\ u(r,L)=V, \tag{9}$$

To solve this problem by separation of variables,9 let u(r, z) = R(r)Z(z) and

$$\frac{1}{rR}\frac{d}{\partial r}\left(r\frac{dR}{dr}\right)=-\frac{1}{Z}\frac{d^2u}{dz^2}=-\frac{k^2}{a^2}, \tag{10}$$

The radial direction only has nontrivial solutions if the separation constant is negative. If so, then we have that

$$\frac{1}{r}\frac{d}{\partial r}\left(r\frac{dR}{dr}\right)+\frac{k^2}{a^2}R=0., \tag{11}$$

The Bessel functions $J_0\left(\frac{kr}{a}\right)$ and$Y_0\left(\frac{kr}{a}\right)$ are the answers to Equation 4. Only $J_0\left(\frac{kr}{a}\right)$ can be a solution because$Y_0\left(\frac{kr}{a}\right)$ goes infinity at $r = 0$. We are forced to select values of k such that $J_0(k)=0$ by the requirement that $u(a,z) = R(a),Z(z) = 0$. Therefore, $J_0\left(\frac{kr}{a}\right)$is the answer in the radial direction, where kn is the nth root of $J_0\left(\frac{kr}{a}\right)=0$. In $z-axis$ direction

$$\frac{d^2Z_n}{dz^2}+\frac{k_n^2}{a^2}Z_n=0, \tag{12}$$

The general solution to Equation 10 is

$$Z_n(z)=A_n\sinh\left(\frac{k_nz}{a}\right)+B_n\cosh\left(\frac{k_nz}{a}\right), \tag{13}$$

Because $u(r,0) = R(r)Z(0) = 0$ and $cosh(0) = 1$, B_n must equal zero. Therefore, the general product solution is

$$u(r,z)=\sum_{n=1}^{\infty}A_nJ_0\left(\frac{k_nr}{a}\right)\sinh\left(\frac{k_nz}{a}\right), \tag{14}$$

The condition that $u(r,L) = V$ determines the arbitrary constant An. Along $z = L$,

$$u(r,L)=V=\sum_{n=1}^{\infty}A_nJ_0\left(\frac{k_nr}{a}\right)\sinh\left(\frac{k_nL}{a}\right), \tag{15}$$

where

$$\sinh\left(\frac{k_nL}{a}\right)A_n=\frac{2V}{a^2J_1^2(k_n)}\int_0^L rJ_0\left(\frac{k_nr}{a}\right)dr, \tag{16}$$

from Equation

$$A_k=\frac{1}{C_k}\int_0^L x\,f(x)J_n(\mu_kL)dx\,, \tag{17}$$

and Equation

$$C_k=\frac{1}{2}L^2J_{n+1}^2(\mu_kL)\,, \tag{18}$$

Thus,

$$\sinh\left(\frac{k_nL}{a}\right)A_n=\frac{2V}{a^2J_1^2(k_n)}\left(\int_0^L rJ_0\left(\frac{k_nr}{a}\right)dr\right), \tag{19}$$

The solution is then

$$u(r,z)=2V\sum_{n=1}^{\infty}\frac{J_0\left(\frac{k_nr}{a}\right)\sinh\left(\frac{k_nz}{a}\right)}{k_nJ_1(k_n)\sinh\left(\frac{k_nL}{a}\right)} \tag{20}$$

The waves near $z = L$ are of great importance. Accordingly, at $r = a$, the solution must make a jump from V to 0. Because of this, the Gibbs phenomenon along this border affects our solution.

The electrostatic potential varies smoothly as we leave that area.

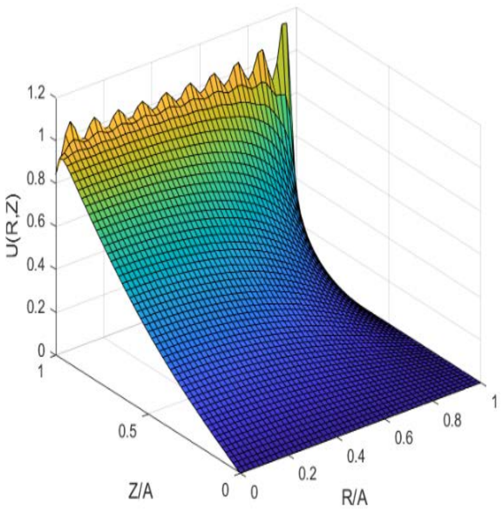

Fig.4 The steady-state potential (divided by V) inside a cylinder with an identical radius and height a, where the top has potential V and the sides and bottom have potential 0.

7. CONCLUSION

The Variational Iteration Method (VIM) combined with the Analytic Solution technique is an effective and efficient way to solve the Laplace equation for steady groundwater flow. The VIM can handle diverse boundary conditions and geometries and produces precise and dependable results. Comparing the VIM with the Analytic Solution shows their strong agreement, validating the VIM's usefulness in resolving the Laplace equation. This method has significant implications for groundwater system management, including designing and optimizing remediation systems, predicting groundwater flow behavior, and evaluating the impact of boundary conditions. The VIM can also be extended to other types of partial differential equations, making it relevant in various domains. Overall, this approach is a valuable addition to existing methods and lays the groundwork for future research in groundwater flow modeling and optimization to improve the sustainable management of groundwater resources.

8. ACKNOWLEDGMENTS

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