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ANALYSIS OF THE STRESS STATE VARIABLES FOR UNSATURATED SOILS WITH COMPRESSIBLE AND INCOMPRESSIBLE SOIL SOLIDS

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ABSTRACT: Net normal stress, $(\sigma - u_a)$ and matric suction, $(u_a - u_w)$ are the stress state variables for unsaturated soils. Stress state variables are the state of stresses in soil that govern the shear strength and the volume change of the soil. Considering the fundamental role of the stress state variables, the correctness of their theoretical derivation is therefore fundamental. In addition, the need to consider the compressibility of the soil solids is recommended in the literature. However, the existing theoretical derivation of stress state variables does not convey a clear explanation for the conditions of compressible and incompressible soil solids. In this paper, a detailed derivation and explanation of the stress state variable for unsaturated soils is presented. The derivation is started from an equilibrium of an unsaturated soil element. The distinction between unsaturated soils with compressible solids and those with incompressible solids is clearly explained. Thus, the derivation in this paper improves the existing theoretical derivation. From the theoretical derivation, the stress state variables for unsaturated soils with incompressible soil solids are the net normal stress, $(\sigma - u_a)$ and the matric suction, $(u_a - u_w)$. For unsaturated soils with compressible soil solids, an additional stress state variable is the pore-air pressure, u_a. The theoretical derivation in this paper better explains the stress state variables for unsaturated soils. This is essential because the net normal stress and the matric suction have been used in various analyses of unsaturated soils.

Keywords: Compressible soil solids, Incompressible soil solids, Stress state variable, Theoretical soil mechanics, Unsaturated soil mechanics

1. INTRODUCTION

Stress state variables are the state of stresses in a soil that govern the mechanical behavior of a soil i.e., shear strength and volume change [1]. The stress state variables should be independent of the physical properties of the soil [1,2]. The stress state variables for unsaturated soils are [1,3]: (i) net normal stress, $(\sigma - u_a)$ and (ii) matric suction, $(u_a - u_w)$. Net normal stress is the stress related to the external load in an unsaturated soil element whereas matric suction is the stress related to the negative pore-water pressure in an unsaturated soil element.

The net normal stress, $(\sigma - u_a)$ and the matric suction, $(u_a - u_w)$ govern the shear strength and volume change of unsaturated soils. The shear strength of unsaturated soils is quantified using the extended Mohr-Coulomb failure envelope equation [1,4]:

$$\tau_{ff} = c' + \left(\sigma_f - u_a\right)_f \tan\phi' + \left(u_a - u_w\right)_f \tan\phi^b \qquad (1)$$

where $(\sigma_f - u_a)_f$ is the net normal stress at failure on the failure plane, $(u_a - u_w)_f$ is the matric suction at failure, and ϕ^b is the angle that quantify the increase in shear strength due to the increase in matric suction.

The volume change of unsaturated soils can be calculated as [1]:

$$de = a_t d\left(\sigma - u_a\right) + a_m d\left(u_a - u_w\right) \qquad (2)$$

where a_t is the coefficient of compressibility with respect to the change in the net normal stress and a_m is the coefficient of compressibility with respect to the change in the matric suction. Both the shear strength (Eq. (1)) and the volume change (Eq. (2)) are governed by the net normal stress, $(\sigma - u_a)$ and the matric suction, $(u_a - u_w)$.

The extended Mohr-Coulomb failure envelope in Eq. (1) has been used in various analyses of shear strength [5-15]. Saing *et al*. [5] performed unconfined compression tests on unsaturated compacted lateritic soil under drying-wetting cycles. The results indicated that the higher the value of matric suction, the higher the value of unconfined compression strength and thus indicating the validity of Eq. (1). Ahmad *et al.* [6] performed unsaturated triaxial tests and found that the deviatoric stresses at failure (i.e., the shear strength) were governed by the matric suction which is also indicating the validity of Eq. (1). Rasool and Kuwano [7] and Ahmad *et al.* [8] performed unsaturated triaxial tests on silt in Japan. The results were plotted using Eq. (1). Tran *et al.* [9], Krisnanto *et al*. [10], and Abeykoon *et al.* [11] performed analyses to obtain the variation of the factor of safety of slope during rainfall in several slopes in Vietnam, Indonesia, and Australia, respectively. Equation (1) was used in the analyses. Do *et al*. [12] performed a probability analysis to obtain the variation of the factor of safety of slope during rainfall in a slope in Vietnam. Hu *et al.* [13] performed slope stability analysis to investigate the effect of the hysteresis SWCC in rainfall-induced slope failure in Japan. Rahardjo *et al.* [14] presented analyses of slope covers to prevent rainfall-induced slope failure. Equation (1) is used in the analyses. A summary of the basic and method of analysis of rainfall-induced slope failure incorporating the net normal stress and the matric suction in Eq. (1) is presented in Krisnanto [15].

The equation for volume change of unsaturated soils (Eq. (2)) has been used in various analyses of volume change [16-23]. Yoshida *et al.* [16] performed swelling tests and heaving calculations of a light industrial building in Canada. The effect of net normal stress (Eq. (2)) in controlling the amount of swelling was obvious. Rahardjo [17] proposed a theory of one-dimensional consolidation for unsaturated soils governed by the net normal stress followed by an experimental verification of the theory by Rahardjo and Fredlund [18]. Abdullahi and Ali [19] used Eq. (2) to calculate volume change due to change in matric suction from root water uptake. Zhang *et al*. [20] performed unsaturated consolidation tests for soils from Japan and found that the void ratio changed due to a change in matric suction as in Eq. (2). Trinh and Tran [21] performed one-dimensional infiltration tests and calculated the amount of heaving using the change in matric suction due to infiltration. Udukumburage [22] performed swelling tests by varying load and water content (or matric suction) and found that the swelling was governed by the normal stress and matric suction. Krisnanto *et al.* [23] Developed predicted volume change constitutive surfaces (using Eq. (2)) for a crushed compacted mudrock.

The shear strength and the volume change are two of the three domains of analysis in geotechnical engineering [24]: (i) shear strength, (ii) volume change, and (iii) flow through porous media. Thus, the net the net normal stress, $(\sigma - u_a)$ and the matric suction, $(u_a - u_w)$ control two of three domains of analysis in geotechnical engineering. This shows the fundamental role of the stress state variables.

Considering the fundamental role of the stress state variables of unsaturated soils, the correctness of their derivation is essential. A derivation that proves that the net normal stress and the matric suction as the stress state variables of unsaturated soils is therefore essential. In addition, the consideration in the derivation for the soils with compressible and incompressible solids is also important. Skempton [25] recommends that an additional stress state variable should be utilized for the soil with compressible soil solids.

This paper presents a detailed derivation of the stress state variables. The derivation in this paper improves the existing theoretical derivation (i.e., in [1,3]). A better explanation is presented on how to obtain the stress state variables for unsaturated soils with compressible soil solids and those with incompressible soil solids.

The main part of this paper is the theoretical derivation with the literature review serves as the background. In the literature review, the existing theoretical derivation is referred. The part from the theoretical derivation that shows the need of a new derivation is highlighted. The derivation is started from the total equilibrium of an unsaturated soil element. The equilibrium for the air, the water, and the contractile skin phases are then calculated. The equilibrium for the soil structure is then calculated from the total equilibrium for the unsaturated soil element and the equilibrium for the air, the water, and the contractile phases. This derivation is performed for the *y*-, *x*-, and *z*- directions. The stress tensor equations are then obtained for the condition of compressible and incompressible soil solids. From these equations, the stress state variables are then extracted.

2. RESEARCH SIGNIFICANCE

In the existing theoretical derivation of stress state variables for unsaturated soils, there is no explicit derivation that explains the condition of the soils with compressible soil solids and incompressible soil solids. In this paper, a detail derivation of the stress state variables for unsaturated soils with compressible and incompressible soil solids is thoroughly explained. Therefore, the derivation in this paper better explains this aspect. Considering the fundamental role of the stress state variables, the correctness of their derivation is essential in theoretical soil mechanics.

3. LITERATURE REVIEW

From [1,3], the stress state variables for unsaturated soils are the net normal stress, $(\sigma - u_a)$ and the matric suction, $(u_a - u_w)$. Both stresses are written in the form of stress tensors as follows [1,3]:

$$\begin{bmatrix} (\sigma_x - u_a) & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & (\sigma_y - u_a) & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & (\sigma_z - u_a) \end{bmatrix} \quad (3)$$

and

$$\begin{bmatrix} (u_a - u_w) & 0 & 0 \\ 0 & (u_a - u_w) & 0 \\ 0 & 0 & (u_a - u_w) \end{bmatrix} \quad (4)$$

The orientations of the normal stresses, τ and the shear stress, σ are shown in Fig. 1. The net normal stress, $(\sigma - u_a)$ and the matric suction, $(u_a - u_w)$ are the stress state variables for unsaturated soils with incompressible soil solids. For unsaturated soils with

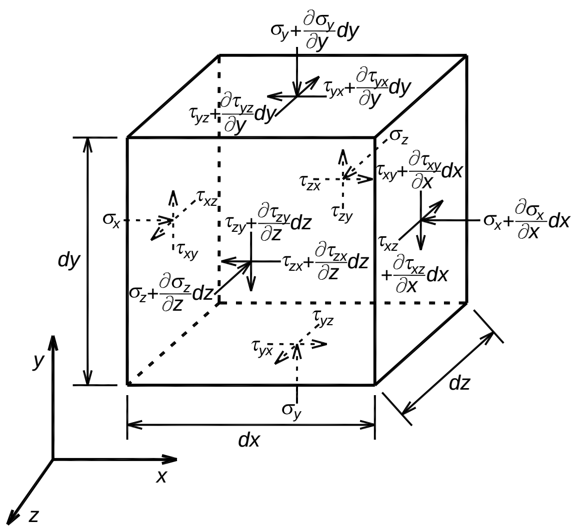

Fig. 1 Soil element and axis orientation in the derivation of the stress state variables for unsaturated soils

compressible soil solids, the stress state variables are the net normal stress, $(\sigma - u_a)$, the matric suction, $(u_a - u_w)$, and the pore-air pressure, u_a [1,3]. In the form of stress tensor, the pore-air pressure, u_a is:

$$\begin{bmatrix} u_a & 0 & 0 \\ 0 & u_a & 0 \\ 0 & 0 & u_a \end{bmatrix} \tag{5}$$

The equilibrium equation in the y-direction is [1,3]:

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial(\sigma_y - u_a)}{\partial y} + (n_w + n_c f^*)\frac{\partial(u_a - u_w)}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + (n_c + n_s)\frac{\partial u_a}{\partial y} + n_s \rho_s g - F_{sy}^w - F_{sy}^a + n_c(u_a - u_w)\frac{\partial f^*}{\partial y} = 0 \tag{6}$$

where τ_{xy} is the shear strength on the x-plane in the y-direction, τ_{xz} is the shear strength on the x-plane in the z-direction, σ_y is the total normal stress in the y-direction, u_a is the pore-air pressure, u_w is the pore-water pressure, $(\sigma_y - u_a)$ is the net normal stress in the y-direction, $(u_a - u_w)$ is the matric suction, n_w is the porosity with respect to the water phase, n_c is the porosity with respect to the contractile skin, n_s is the porosity with respect to the soil solid, ρ_s is the soil solid density, g is the gravitational acceleration, F_{sy}^w is the interaction force (i.e., body force) between the water phase and the soil solid in the y-direction, F_{sy}^a is the interaction force (i.e., body force) between the air phase and the soil solid in the y-direction, and f^* is the final interaction between the contractile skin and the soil structure equilibrium. The axis orientation for the derivation is shown in Fig. 1. The porosities are defined as follows [1]:

$$n_a = V_a / V \tag{7}$$

$$n_w = V_w / V \tag{8}$$

$$n_c = V_c / V \tag{9}$$

$$n_s = V_s / V \tag{10}$$

$$n_a + n_w + n_c + n_s = 1 \tag{11}$$

where V is the total volume of soil, V_a is the volume of air, V_w is the volume of water, V_c is the volume of contractile skin, and V_s is the volume of solid. The volumes used in Eqs. (7) to (10) are shown in the rigorous four phases diagram of unsaturated soils (Fig. 2). Rigorous four phases diagram is proposed by [1] to quantify all four phases in unsaturated soils.

The equilibrium equation in the x- and z-directions are [1,3]:

$$\frac{\partial(\sigma_x - u_a)}{\partial x} + (n_w + n_c f^*)\frac{\partial(u_a - u_w)}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + (n_c + n_s)\frac{\partial u_a}{\partial x} - F_{sx}^w - F_{sx}^a + n_c(u_a - u_w)\frac{\partial f^*}{\partial x} = 0 \tag{12}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial(\sigma_z - u_a)}{\partial z} + (n_w + n_c f^*)\frac{\partial(u_a - u_w)}{\partial z} + (n_c + n_s)\frac{\partial u_a}{\partial z} - F_{sz}^w - F_{sz}^a + n_c(u_a - u_w)\frac{\partial f^*}{\partial z} = 0 \tag{13}$$

The stress tensors of stress state variables (Eqs. (3) to (5)), are extracted from the equilibrium equation (Eqs. (6), (12), and (13)).

Equations (6), (12), and (13) use the pore-air pressure, u_a as the stress reference. This means that the stress differences between σ and u_a and between u_a and u_w are used in the equilibrium equation. This results in the stresses $(\sigma - u_a)$ and $(u_a - u_w)$. From these equations, the stress state variables in Eqs. (3) to (5) are obtained. For the soil with incompressible solids, the stress state variable u_a in Eqs. (6), (12), and (13) can be eliminated [1]. Thus, the stress state variables are Eqs. (3) and (4).

From Eq. (11):

$$n_s = 1 - n_a - n_w - n_c \tag{14}$$

The contractile skin plays an important role in mechanical behavior of unsaturated soils. However, in terms of the volume-mass of soils, the volume of contractile skin, V_c is small as compared to the volume of soil solids and the volume of voids [1]. Also, the thickness of contractile skin is in the order of magnitude of 10^{-7} cm [26], which is very small as compared to the soil grain size. Therefore, in the calculation of the volume-mass of soils, the rigorous four phases diagram (Fig. 2) can be simplified as the simplified three phases diagram (Fig. 3) and the porosity in terms of the soil solids, n_c can be assumed as zero. Thus Eq. (14) becomes:

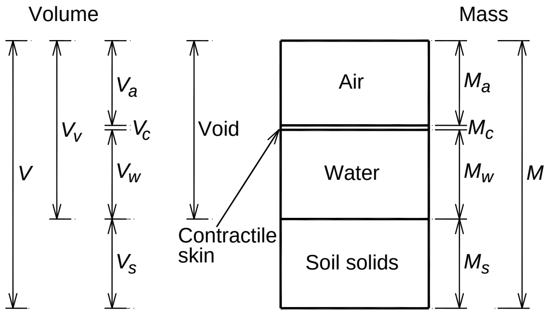

Fig. 2 Rigorous four phases diagram

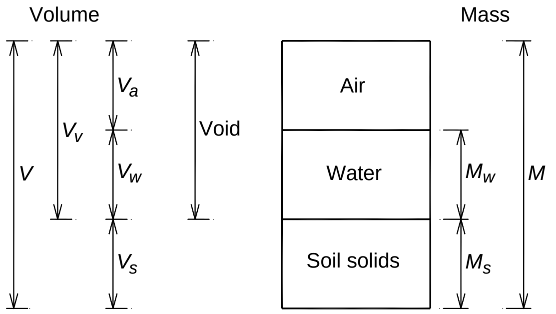

Fig. 3 Simplified three phases diagram

$$n_s = 1 - n_a - n_w \tag{15}$$

$$n_s = 1 - n \tag{16}$$

From basic soil mechanics:

$$n = V_v / V \tag{17}$$

where n is the porosity and V_v is the volume of void.

The porosity, n is in the range of 0.12 to 0.84 as shown in Tables 1 and 2. Using these values and Eq. (16), the porosity in terms of the soil solids, n_s is in the range of 0.88 to 0.16. Considering this range of value of n_s, the term $(n_c + n_s)\partial u_a/\partial y$ in Eqs. (6), (12), and (13) is not zero. This means that the pore-air pressure, u_a cannot be eliminated from Eqs. (6), (12), and (13). A reconsideration of the derivation is therefore needed to investigate the stress state variables of unsaturated soils (Eqs. (3) to (5)).

Table 1. The values of porosity of several types of soils (data from [27])

Soil Type	Porosity, n
Uniform sand, loose	0.46
Uniform sand, dense	0.34
Mixed-grained sand, loose	0.40
Mixed-grained sand, dense	0.30
Glacial till, very mixed-grained	0.20
Soft glacial clay	0.55
Stiff glacial clay	0.37
Soft slightly organic clay	0.66
Soft very organic clay	0.75
Soft bentonite	0.84

Table 2. The values of porosity of several granular soils (data from [28,29])

Granular Soil Type	Porosity, n	
	n_{max} (loose)	n_{min} (dense)
Uniform material: equal spheres	0.48	0.26
Uniform material: standard Ottawa sand	0.44	0.33
Uniform material: clean uniform sand (fine or medium)	0.50	0.29
Uniform material: uniform, inorganic silt	0.52	0.29
Well-graded materials: silty sand	0.47	0.23
Well-graded materials: clean, fine to coarse sand	0.49	0.17
Well-graded materials: micaceous sand	0.55	0.29
Well-graded materials: silty sand and gravel	0.46	0.12

4. THEORETICAL DERIVATION

4.1 Derivation in the *y*-Direction

The axis orientation for the derivation follows that in Fig. 1.

4.1.1 Total equilibrium of unsaturated soil element

Sum of forces in the y-direction [1]:

$$\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho g\right) dx.dy.dz = \rho \frac{Dv_y}{Dt} \tag{18}$$

where ρ is the total density of the soil, g is the gravitational acceleration, and dx, dy, dz are the dimension of the element in the x-, y-, and z-directions, respectively. Because the soil element is in static condition, the right term is zero. Therefore, Eq. (18) becomes:

$$\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho g\right) dx.dy.dz = 0 \tag{19}$$

4.1.2 Independent phase equilibrium

From the relationship of density:

$$\rho = (M_a + M_w + M_c + M_s)/V \tag{20}$$

From the definition of porosities (Eqs. (7) to (11)):

$$\rho = n_a\rho_a + n_w\rho_w + n_c\rho_c + n_s\rho_s \tag{21}$$

where M_a is the mass of the air phase, M_w is the mass of the water phase, M_c is the mass of the contractile skin, and M_s is the mass of the soil solid. In this study, it is assumed that:

$$a_a + a_w + a_c + a_s = 1 \tag{22}$$

where a_a is the cross-sectional area filled with air per gross cross-sectional area of the soil element, a_w is the cross-sectional area filled with water per gross cross-sectional area of the soil element, a_c is the cross-sectional area filled with contractile skin per gross cross-sectional area of the soil element, and a_s is the cross-sectional area filled with solid per gross cross-sectional area of the soil element. For unsaturated soils with compressible soil solids, Eq. (22) becomes:

$$a_a + a_w + a_c + a_s + \Delta a_s = 1 \tag{23}$$

where Δa_s is the additional cross-sectional area filled with solid per gross cross-sectional area of the soil element due to the compressibility of the soil solids.

4.1.3 Equilibrium for the water phase

Sum of forces in the y-direction equal to zero:

$$\begin{aligned}&-a_w\left(u_w + \frac{\partial u_w}{\partial y}dy\right)dx.dz + a_w u_w dx.dz\\&-n_w\rho_w g dx.dy.dz - F_{sy}^w dx.dy.dz - F_{cy}^w dx.dy.dz = 0\end{aligned} \tag{24}$$

where F_{cy}^w is the interaction force (i.e., body force) between the water phase and the contractile skin in the y-direction. Further derivation results in:

$$\left(a_w\frac{\partial u_w}{\partial y} + n_w\rho_w g + F_{sy}^w + F_{cy}^w\right)dx.dy.dz = 0 \tag{25}$$

4.1.4 Equilibrium for the air phase

Sum of forces in the y-direction equal to zero:

$$\begin{aligned}&-a_a\left(u_a + \frac{\partial u_a}{\partial y}dy\right)dx.dz + a_a u_a dx.dz\\&-n_a\rho_a g dx.dy.dz - F_{sy}^a dx.dy.dz - F_{cy}^a dx.dy.dz = 0\end{aligned} \tag{26}$$

where F_{cy}^a is the interaction force (i.e., body force) between the air phase and the contractile skin in the y-direction. Further derivation results in:

$$\left(a_a\frac{\partial u_a}{\partial y} + n_a\rho_a g + F_{sy}^a + F_{cy}^a\right)dx.dy.dz = 0 \tag{27}$$

4.1.5 Equilibrium for the contractile skin

Sum of forces in the y-direction equal to zero:

$$\begin{aligned}&a_c\left(\sigma_y^c + \frac{\partial f^*}{\partial y}(u_a - u_w)dy + f^*\frac{\partial(u_a - u_w)}{\partial y}dy\right)dx.dz\\&-a_c\sigma_y^c dx.dz - n_c\rho_c g + F_{cy}^w dx.dy.dz + F_{cy}^a dx.dy.dz = 0\end{aligned} \tag{28}$$

Further derivation results in:

$$\left(\begin{aligned}&-a_c(u_a - u_w)\frac{\partial f^*}{\partial y} - a_c f^*\frac{\partial(u_a - u_w)}{\partial y}\\&+n_c\rho_c g - F_{cy}^w - F_{cy}^a\end{aligned}\right)dx.dy.dz = 0 \tag{29}$$

4.1.6 Equilibrium for the soil structure

From [1]:

total equilibrium of unsaturated soil element
− equilibrium for water phase
− equilibrium for air phase
− equilibrium for contractile skin = 0 (30)

$$\text{Eq. }(19) - \text{Eq. }(25) - \text{Eq. }(27) - \text{Eq. }(29) = 0 \tag{31}$$

$$\begin{aligned}&\left(\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} + \rho g\right)dx.dy.dz\\&-\left(a_w\frac{\partial u_w}{\partial y} + n_w\rho_w g + F_{sy}^w + F_{cy}^w\right)dx.dy.dz\\&-\left(a_a\frac{\partial u_a}{\partial y} + n_a\rho_a g + F_{sy}^a + F_{cy}^a\right)dx.dy.dz\\&-\left(\begin{aligned}&-a_c\frac{\partial f^*}{\partial y}(u_a - u_w) - a_c f^*\frac{\partial(u_a - u_w)}{\partial y}\\&+n_c\rho_c g - F_{cy}^w - F_{cy}^a\end{aligned}\right)dx.dy.dz = 0\end{aligned} \tag{32}$$

Further derivation results in:

$$\begin{aligned}&\left(\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} + \rho g\right)dx.dy.dz\\&-\left(a_w\frac{\partial u_w}{\partial y} + n_w\rho_w g + F_{sy}^w + F_{cy}^w\right)dx.dy.dz\\&-\left(a_a\frac{\partial u_a}{\partial y} + n_a\rho_a g + F_{sy}^a + F_{cy}^a\right)dx.dy.dz\\&-\left(\begin{aligned}&-a_c\frac{\partial f^*}{\partial y}(u_a - u_w) - a_c f^*\frac{\partial(u_a - u_w)}{\partial y}\\&+n_c\rho_c g - F_{cy}^w - F_{cy}^a\end{aligned}\right)dx.dy.dz = 0\end{aligned} \tag{33}$$

Divide both sides with $dx.dy.dz$ and continue the derivation results in:

$$\begin{aligned}&\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} - a_a\frac{\partial u_a}{\partial y} - a_w\frac{\partial u_w}{\partial y} + a_c f^*\frac{\partial(u_a - u_w)}{\partial y}\\&+\frac{\partial\tau_{zy}}{\partial z} + (\rho - n_a\rho_a - n_w\rho_w - n_c\rho_c)g - F_{sy}^w - F_{sy}^a\\&+a_c(u_a - u_w)\frac{\partial f^*}{\partial y} = 0\end{aligned} \tag{34}$$

From Eq. (21):

$$n_s\rho_s = \rho - n_a\rho_a - n_w\rho_w - n_c\rho_c \tag{35}$$

Substituting Eq. (35) into Eq. (34) results in:

$$\begin{aligned}&\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} - a_a\frac{\partial u_a}{\partial y} - a_w\frac{\partial u_w}{\partial y}\\&+a_c f^*\frac{\partial(u_a - u_w)}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} + n_s\rho_s g\\&-F_{sy}^w - F_{sy}^a + a_c(u_a - u_w)\frac{\partial f^*}{\partial y} = 0\end{aligned} \tag{36}$$

In this derivation the pore-air pressure, u_a is used as the stress reference. From Eq. (23):

$$a_a = 1 - a_w - a_c - a_s - \Delta a_s \tag{37}$$

Substituting Eq. (37) into Eq. (36) and continue the derivation results in:

$$\begin{aligned}&\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial(\sigma_y - u_a)}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} + (a_w + a_c f^*)\frac{\partial(u_a - u_w)}{\partial y}\\&+(a_c + a_s + \Delta a_s)\frac{\partial u_a}{\partial y} + n_s\rho_s g - F_{sy}^w - F_{sy}^a\end{aligned}$$

$$+a_c\left(u_a-u_w\right)\frac{\partial f^*}{\partial y}=0 \tag{38}$$

It is assumed that:

$$a_c=f\left(n_c\right) \tag{39}$$

and

$$a_w=f\left(n_w\right) \tag{40}$$

In addition, since the volume of contractile skin is small:

$$n_c\approx 0 \tag{41}$$

Therefore:

$$f\left(n_c\right)\approx 0 \tag{42}$$

From [25], [28], and [29]:

$$a_s\approx 0 \tag{43}$$

Substituting Eqs. (39), (40), (42), and (43) into Eq. (38) results in:

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial\left(\sigma_y-u_a\right)}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}+f\left(n_w\right)\frac{\partial\left(u_a-u_w\right)}{\partial y}+\Delta a_s\frac{\partial u_a}{\partial y}+n_s\rho_s g-F_{sy}^w-F_{sy}^a=0 \tag{44}$$

If the soil solids are incompressible, then:

$$\Delta a_s=0 \tag{45}$$

For unsaturated soils with incompressible solids, by substituting Eq. (45) into Eq. (44) results in the following relationship:

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial\left(\sigma_y-u_a\right)}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}+f\left(n_w\right)\frac{\partial\left(u_a-u_w\right)}{\partial y}+n_s\rho_s g-F_{sy}^w-F_{sy}^a=0 \tag{46}$$

4.2 Derivation in the *x*-Direction

Using the same derivation method from Eq. (18) to (46) results in the following relationship:

$$\frac{\partial\left(\sigma_x-u_a\right)}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+f\left(n_w\right)\frac{\partial\left(u_a-u_w\right)}{\partial x}+\Delta a_s\frac{\partial u_a}{\partial x}-F_{sx}^w-F_{sx}^a=0 \tag{47}$$

For unsaturated soils with incompressible solids, by substituting Eq. (45) into Eq. (47) results in the following relationship:

$$\frac{\partial\left(\sigma_x-u_a\right)}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+f\left(n_w\right)\frac{\partial\left(u_a-u_w\right)}{\partial x}-F_{sx}^w-F_{sx}^a=0 \tag{48}$$

4.3 Derivation in the *z*-Direction

Using the same derivation method from Eq. (18) to (46) results in the following relationship:

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial\left(\sigma_z-u_a\right)}{\partial z}+f\left(n_w\right)\frac{\partial\left(u_a-u_w\right)}{\partial z}+\Delta a_s\frac{\partial u_a}{\partial z}-F_{sz}^w-F_{sz}^a=0 \tag{49}$$

For unsaturated soils with incompressible solids, by substituting Eq. (45) into Eq. (49) results in the following relationship:

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial\left(\sigma_z-u_a\right)}{\partial z}+f\left(n_w\right)\frac{\partial\left(u_a-u_w\right)}{\partial z}-F_{sz}^w-F_{sz}^a=0 \tag{50}$$

For unsaturated soils with compressible solids, Eqs. (44), (47), and (49) can be written in the following stress tensor equation:

$$\begin{bmatrix}(\sigma_x-u_a) & \tau_{yx} & \tau_{zx}\\ \tau_{xy} & (\sigma_y-u_a) & \tau_{zy}\\ \tau_{xz} & \tau_{yz} & (\sigma_z-u_a)\end{bmatrix}\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{Bmatrix}+f\left(n_w\right)\begin{bmatrix}(u_a-u_w) & 0 & 0\\ 0 & (u_a-u_w) & 0\\ 0 & 0 & (u_a-u_w)\end{bmatrix}\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{Bmatrix}+\Delta a_s\begin{bmatrix}u_a & 0 & 0\\ 0 & u_a & 0\\ 0 & 0 & u_a\end{bmatrix}\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{Bmatrix}+\begin{Bmatrix}0\\ n_s\rho_s g\\ 0\end{Bmatrix}-\begin{Bmatrix}F_{sx}^w\\ F_{sy}^w\\ F_{sz}^w\end{Bmatrix}-\begin{Bmatrix}F_{sx}^a\\ F_{sy}^a\\ F_{sz}^a\end{Bmatrix}=0 \tag{51}$$

From Eq. (51), the stress tensors for unsaturated soils with compressible solids are:

$$\begin{bmatrix}(\sigma_x-u_a) & \tau_{yx} & \tau_{zx}\\ \tau_{xy} & (\sigma_y-u_a) & \tau_{zy}\\ \tau_{xz} & \tau_{yz} & (\sigma_z-u_a)\end{bmatrix} \tag{52}$$

and

$$\begin{bmatrix}(u_a-u_w) & 0 & 0\\ 0 & (u_a-u_w) & 0\\ 0 & 0 & (u_a-u_w)\end{bmatrix} \tag{53}$$

and

$$\begin{bmatrix}u_a & 0 & 0\\ 0 & u_a & 0\\ 0 & 0 & u_a\end{bmatrix} \tag{54}$$

For unsaturated soils with incompressible solids, Eqs. (46), (48), and (50) can be written in the following stress tensor equation:

$$\begin{bmatrix}(\sigma_x-u_a) & \tau_{yx} & \tau_{zx}\\ \tau_{xy} & (\sigma_y-u_a) & \tau_{zy}\\ \tau_{xz} & \tau_{yz} & (\sigma_z-u_a)\end{bmatrix}\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{Bmatrix}+f\left(n_w\right)\begin{bmatrix}(u_a-u_w) & 0 & 0\\ 0 & (u_a-u_w) & 0\\ 0 & 0 & (u_a-u_w)\end{bmatrix}\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{Bmatrix}$$

$$+\begin{Bmatrix} 0 \\ n_s \rho_s g \\ 0 \end{Bmatrix} - \begin{Bmatrix} F_{sx}^w \\ F_{sy}^w \\ F_{sz}^w \end{Bmatrix} - \begin{Bmatrix} F_{sx}^a \\ F_{sy}^a \\ F_{sz}^a \end{Bmatrix} = 0 \quad (55)$$

From Eq. (55), the stress tensors for unsaturated soils with incompressible solids are Eqs. (52) and (53). The stresses σ, u_a, and u_w in an unsaturated soil element is shown in Fig. 4. The continuum soil element of unsaturated soil with the stress state variables using u_a as the stress reference is shown in Fig. 5.

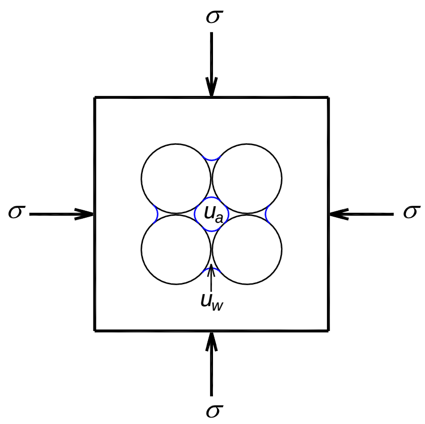

Fig. 4 The stresses σ, u_a, and u_w in an unsaturated soil element

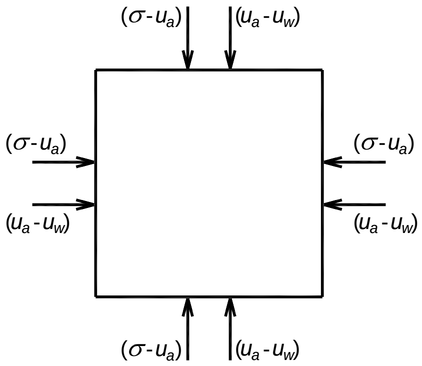

Fig. 5 Continuum element of unsaturated soil with the stress state variables using u_a as the stress reference

5. DISCUSSIONS

A comparison between the results of the theoretical derivation of the stress state variables in this study (Eqs. (51) and (55)) with those from the literature (Eqs. (6), (12), and (13)) shows that the derivation in this study performs better to explain the elimination of the pore-air pressure, u_a as the soil solids is incompressible. In this study, the compressibility of the soil is represented by the parameter Δa_s in the term $\Delta a_s \, \partial u_a / \partial y$ (Eq. (51)) whereas in the literature (Eqs. (6), (12), and (13)), the compressibility is represented by the parameter $(n_c + n_s)$ in the term $(n_c + n_s)\partial u_a / \partial y$. It is easier to accept that the term $\Delta a_s \, \partial u_a / \partial y$ in Eq. (51) becomes zero for incompressible soil solids (see also Eq. (45)) than the term $(n_c + n_s)\partial u_a / \partial y$ becomes zero. This comes from the fact that the value of the void ratio in terms of the soil solids, n_s is not zero, as indicated in Eqs. (16), (17), Tables 1 and 2.

The term $\Delta a_s \, \partial u_a / \partial y$ resulting from the theoretical derivation in this study as compared to the term $(n_c + n_s)\partial u_a / \partial y$ means that this study better proves the validity of the net normal stress, ($\sigma - u_a$), matric suction, ($u_a - u_w$) as the stress state variables of unsaturated soils. This proof is essential because the net normal stress and the matric suction have been used in various analyses of shear strength [5-15] and volume change [16-23]. Both are in theoretical, laboratory, and engineering practice fields as explained in the introduction. Thus, any weakness in the conclusion of the stress state variables results in the invalidity of the existing analyses.

For the assumption in Eq. (43), the quantification of the cross-sectional area filled with solid per gross cross- sectional area of the soil element is shown in the representative elementary volume (REV) in Figs. 6 and 7. The axis orientation in Figs. 6 and 7 complies with the plane a_s in Eq. (38). For the x- and y-directions, the planes are shown in Fig. 6 whereas for the y- and z-directions are shown in Fig. 7. In coarse-grained and fine-grained cohesive soils, the value of a_s is small [25,28,29]. This gives the basis of the assumption in Eq. (43).

Another assumption is that in Eq. (41). The thickness of contractile skin is in the order of magnitude of 10^{-7} cm [26], which is very small as compared to the soil grain size. In addition, Fredlund and Rahardjo [1] indicates that the volume of contractile skin, V_c is small compared to the volume of soil solids and the volume of voids. These provide a basis of the assumption in Eq. (41).

6. CONCLUSIONS

Literature review (Eqs. (6), (12), (13), (16), (17), Tables 1 and 2) shows that there is a difficulty in obtaining net normal stress, ($\sigma - u_a$) and matric suction, ($u_a - u_w$) as the stress state variable for unsaturated soils with incompressible soil solids. This is because the value of the void ratio in terms of the soil solids, n_s is not zero, as indicated in Eqs. (16), (17), Tables 1 and 2. Therefore, the term $(n_c + n_s)\partial u_a / \partial y$ in Eqs. (6), (12), and (13) is not zero. This means that the pore-air pressure, u_a cannot be eliminated from Eqs. (6), (12), and (13).

A comparison between the results of theoretical derivation of the stress state variables in this study (Eqs. (51) and (55)) with that from the literature (Eqs.

(6), (12), and (13)) shows that the derivation in this study performs better to explain the elimination of the pore-air pressure, u_a as the soil solids is incompressible. The term $\Delta a_s \, \partial u_a / \partial y$ resulting from this study as compared to the term $(n_c + n_s) \partial u_a / \partial y$ from the existing derivation shows that the derivation in this study better explains that for the soil with compressible solids, the stress state variable u_a can be eliminated. This means that this study better proves the validity of the net normal stress, $(\sigma - u_a)$, matric suction, $(u_a - u_w)$ as the stress state variables of unsaturated soils with incompressible solids.

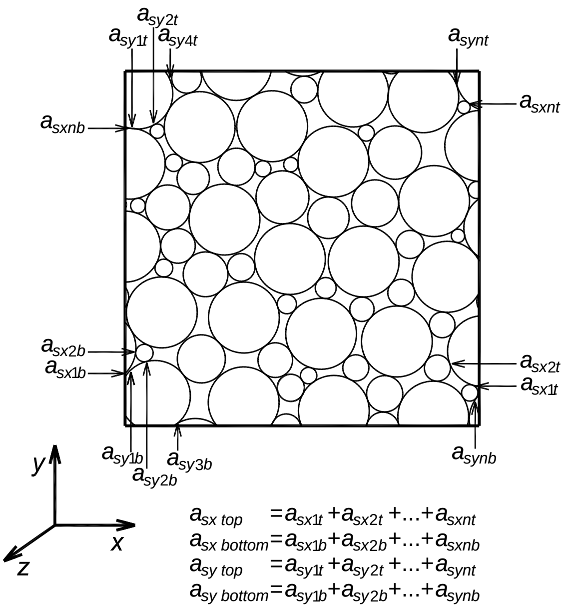

Fig. 6 A representative elementary volume (REV) with a_s in x- and y- planes

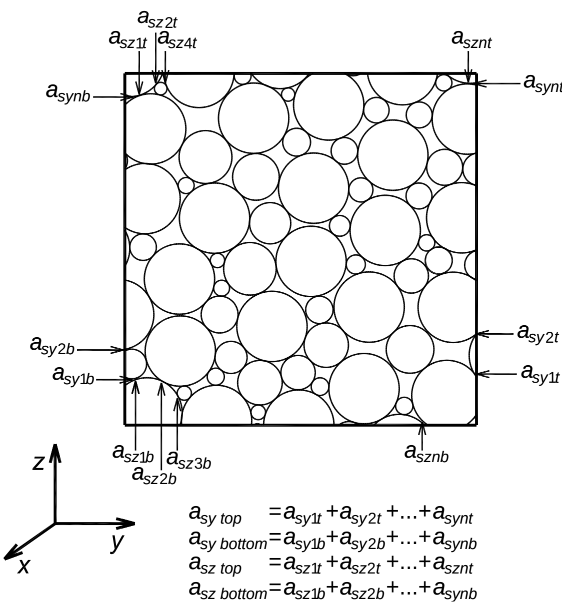

Fig. 7 A representative elementary volume (REV) with a_s in y-z plane

This study shows that the stress state variables for unsaturated soils with compressible soil solids using pore-air pressure, u_a as the reference are: (i) net normal stress, $(\sigma - u_a)$, (ii) matric suction, $(u_a - u_w)$, and (iii) pore-air pressure, u_a. The stress state variables for unsaturated soils with incompressible soil solids are: (i) net normal stress, $(\sigma - u_a)$ and (ii) matric suction, $(u_a - u_w)$.

The theoretical derivation proves the validity of net normal stress, $(\sigma - u_a)$ and the matric suction, $(u_a - u_w)$ as the stress state variables for unsaturated soils with incompressible soil solids. This is essential because the net normal stress and the matric suction have been used in various analyses of shear strength and volume change, in theoretical and engineering practice domains. Therefore, the validity of these analyses depends on the validity of the theoretical derivation of the stress state variables.

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