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AN IMPROVED METHOD OF CALCULATING BEAM DEFORMATION CONSIDERING TRANSVERSE SHEAR STRAIN

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ABSTRACT: In 1921, Timoshenko improved upon the Euler–Bernoulli beam theory by adding the effect of transverse shear strains. In the Euler–Bernoulli beam theory, transverse shear strains are neglected, and cross-sections remain plane and perpendicular to the neutral axis. In the Timoshenko beam theory, cross-sections still remain plane but are no longer perpendicular to the neutral axis. Although the transverse shear strain approaches zero, the Timoshenko beam theory does not converge on the Euler–Bernoulli beam theory due to the shear locking. Many authors have attempted to overcome this problem and although they have brought acceptable results, these still have theoretical limitations. Therefore, in this paper, the author presents an improved method of calculating the beam considering transverse shear strain to overcome the theoretical limitations of previous authors. The problem is addressed by the use of numerical methods which achieve convergence and avoid the issue of shear locking.

Keywords: Euler–Bernoulli beam theory, Timoshenko beam theory, Transverse shear strain, Shear locking

1. INTRODUCTION

In the Euler–Bernoulli beam theory, transverse shear strains are neglected, whereas in the Timoshenko beam theory, transverse shear strains are considered. In Euler–Bernoulli beams, the cross-section remains perpendicular to the neutral axis after bending, whereas in Timoshenko beams initially the cross-section is perpendicular to the neutral axis but it does not remain perpendicular after bending [1]. These relations are shown in Fig.1.

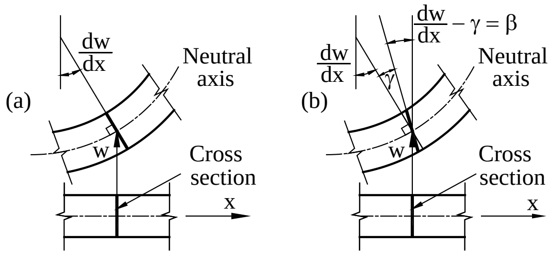

Fig.1 Deformation of cross-section: (a) Euler–Bernoulli beams; (b) Timoshenko beams

According to Timoshenko beam theory, we have:

$$\frac{dw}{dx} = \beta + \gamma \tag{1}$$

where:

w is the deflection of the neutral axis of the beam (vertical displacement);

β is the rotation of the cross-section plane (caused by bending moment);

γ is the transverse shear strain (caused by shear force).

There is a long history of research on this issue, for example, by Wilson [2,3], Zienkiewicz [4], and Bathe [5]. Based on the theory considering the above transverse shear strain, the authors often use finite element methods to propose numerical solutions to the problem. When using the finite element method, the authors used two independent elements to correspond with the two unknowns w and β. The element type only has displacement and a rotation angle caused by the moment to describe the bending element. However, as the transverse shear strain approaches zero, these beam theories do not converge on the Euler–Bernoulli beam theory due to the shear locking.

Wilson is a professor emeritus of the University of California at Berkeley, USA, and one of the first authors to propose a solution to the shear locking when calculating beams considering the transverse shear strain. He first introduced incompatible displacements into rectangular isoparametric finite elements at a conference in 1971, where the method was received with great skepticism by fellow researchers [2,3]. However, the results for both displacements and stresses for rectangular elements were very close to the results from the nine-node isoparametric element of other researchers such as Zienkiewicz [4] and Bathe [5].

Assessing the method proposed by Professor Wilson and other scientists to reduce the shear locking, Mathematics Professor Strang of MIT commented: "Two theoretical crimes committed were displacement compatibility was violated and the method was not verified with examples using non-rectangular elements" [6].

In the 3rd edition in 2002 of reference [3], Professor Wilson used a cubic polynomial to represent interpolation functions to ensure compatibility conditions and this was applied to popular commercial software for calculating structures such as SAP-2000, ETABS, and SAFE.

A beam element considering transverse shear strain according to Wilson is shown in Fig.2.

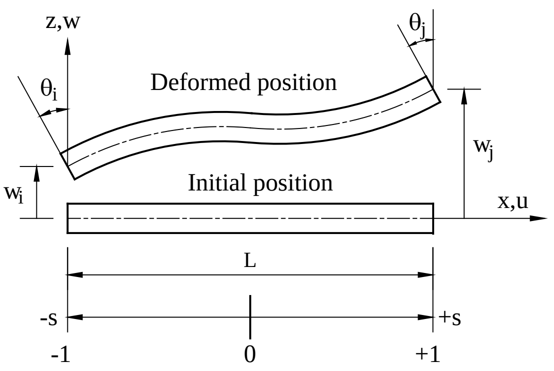

Fig.2 Typical beam element with shear strain

To maintain a consistent assumption for cubic normal displacement, Wilson had to add a constant shear strain along the element boundary. Therefore, he also advises not to use a cubic polynomial to represent the displacement for non-rectangular elements because, for non-rectangular elements with variable cross-section height, the shear strain condition is not correct along the element boundary.

Vu Thanh Thuy (2010), when studying the internal force and displacement of a bending beam considering the influence of transverse shear strain, proposed to use the two functions of displacement and shear force as two independent unknowns to develop and solve the bending beam problem considering transverse shear strain [7]. Vu Thanh Thuy solved some of the bending beam problems in the analytical method by using the extreme Gauss's principle method (proposed by Professor Ha Huy Cuong [8]) with optimization of the parameters of the polynomial representing the displacement implicit functions and the shear force.

In this study, the effects of transverse shear strain on the internal force, displacement and deformation of the bending beam are investigated by an improved method. The results of calculation are compared with the model calculated using commercial software of which Professor Wilson is one of the authors, and the results are calculated according to the analytical method of Vu Thanh Thuy.

2. RESEARCH SIGNIFICANCE

In this paper, the author presents an improved method of calculating the beam considering transverse shear strain to overcome the theoretical limitations of previous authors. The problem is addressed using the finite element method, which achieves convergence and avoids the issue of shear locking. In particular, considering transverse shear strain, we get a redistribution of the internal force in the cantilever and simply supported beam subjected to uniformly distributed load.

3. AN IMPROVED METHOD OF CALCULATING BEAM

3.1 Shape Functions for Beam Element

When not considering the transverse shear strain, to describe the bending beam, it is usual to choose elements with 4 parameters at the nodes, using the cubic polynomial (with 4 parameters) to represent the displacement of the bending beam element. Four cubic polynomial parameters are determined by the displacement and rotation angle at both ends of the element, and must satisfy the equilibrium interpolation condition. Since we choose the displacement function of the cubic polynomial, the force acting on the beam must be the concentrated force located at the nodes of the element. The isoparametric beam element is shown in Fig.3.

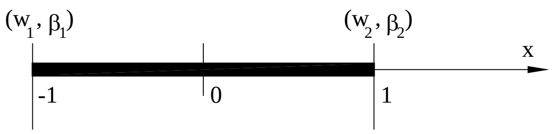

Fig.3 Isoparametric beam element not considering transverse shear strain

The displacement function of the bending beam element as a cubic polynomial is:

$$w(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$$
$$= \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = [U]\{\alpha\} \tag{2}$$

It is possible to find four parameters including the displacement values and rotation angle at both ends by replacing the node coordinates with the displacement function and its derivative, respectively:

Node 1 (x = -1):

$$w_1 = w(-1) = \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 \tag{3}$$

$$\beta_1 = \left.\frac{dw}{dx}\right|_{-1} = 0 + \alpha_2 - 2\alpha_3 + 3\alpha_4 \tag{4}$$

Node 2 (x = 1):

$$w_1 = w(+1) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \tag{5}$$

$$\beta_2 = w'(1) = 0 + \alpha_2 + 2\alpha_3 + 3\alpha_4 \tag{6}$$

The matrix form can be written as follows:

$$\{X\} = \begin{bmatrix} y_1 \\ \beta_1 \\ y_2 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = [C]\{\alpha\} \tag{7}$$

Therefore,

$$\{\alpha\} = [C]^{-1}\{X\} \tag{8}$$

Substituting this value $\{\alpha\}$ into Eq. (2), we have:

$$w = [U]\{\alpha\} = [U][C]^{-1}\{X\} = [f]\{X\} \tag{9}$$

The $[f] = [U][C]^{-1}$ can be determined from the inverse of the [C] matrix. However, the problem is realized by a Matlab code. The result of $[f]$ can be written as:

$$[f] = [f_1 \quad f_2 \quad f_3 \quad f_4] \tag{10}$$

where:

$$f_1 = \frac{1}{4}(x-1)^2(x+2)$$

$$f_2 = \frac{1}{4}(x-1)^2(x+1)$$

$$f_3 = \frac{1}{4}(x+1)^2(2-x)$$

$$f_4 = \frac{1}{4}(x+1)^2(x-1)$$

Therefore, when we know the displacement and rotation angle at both ends of the element, the displacement at each point in the element is determined through the function

$$w = (f_1 w_1 + f_2\beta_1 + f_3 w_2 + f_4\beta_2) \tag{11}$$

The rotation angle at each point in an element is determined through the function

$$\beta = \frac{dy}{dx} \tag{12}$$

According to Eq. (25) and Eq. (26), we have

$$w|_{x=-1} = w_1;\ w|_{x=1} = w_2;\ \left.\frac{dw}{dx}\right|_{x=-1} = \beta_1;\ \left.\frac{dw}{dx}\right|_{x=1} = \beta_2$$

Thus, the interpolation functions are compatible between the mathematics and mechanics and the elements will be compatible elements that ensure convergence conditions to exact solutions.

3.2 Selection of the Bending Beam Element Considering Transverse Shear Strain

When considering the transverse shear strain, it is necessary to define the two independent unknowns of the displacement w and shear force V. Therefore, when applying the finite element method to solve the problem of the bending beam, it is necessary to choose two elements: the displacement element and shear force element, using the cubic polynomial interpolation function for the displacement isoparametric element to represent the displacement of the bending beam. The cubic polynomial function has four parameters that are determined through two displacement values and its first derivative at the node of the element (Fig.4).

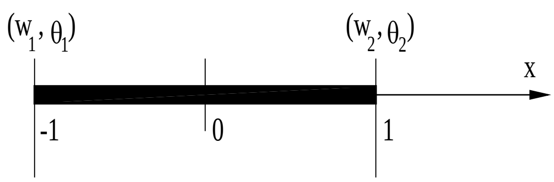

Fig.4 Displacement isoparametric element considering transverse shear strain

The displacement at each point in the element is determined through the function

$$w = (f_1 w_1 + f_2\beta_1 + f_3 w_2 + f_4\beta_2) \tag{13}$$

where f_1, f_2, f_3, f_4 are cubic polynomials that are determined similarly when not considering transverse shear strain.

From the first derivative of the displacement function, we get the total rotation angle at each point in the element determined through the function

$$\theta = \frac{dw}{dx} \tag{14}$$

According to Eq. (13) and Eq. (14), we have

$$w\big|_{x=-1} = w_1;\ \ w\big|_{x=1} = w_2;\ \ \frac{dw}{dx}\Big|_{x=-1} = \theta_1;\ \ \frac{dw}{dx}\Big|_{x=1} = \theta_2$$

Using the quadratic polynomial to represent the shear force, three parameters of the function will be determined through three shear force values at the three nodes of the beam element. The shear isoparametric element is shown in Fig.5.

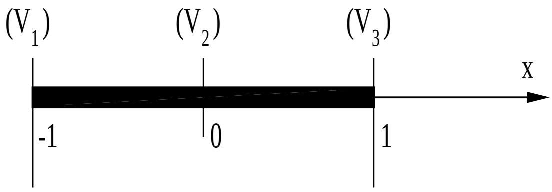

Fig.5 Shear force isoparametric element

The shear force at each point in the element is determined through the function

$$Q = f_5Q_1 + f_6Q_2 + f_7Q_3 \tag{15}$$

where f_5, f_6, f_7 are quadratic polynomials that are determined similarly to the displacement, as

$$f_5 = \frac{1}{2}x(x-1) \tag{16}$$

$$f_6 = 1 - x^2 \tag{17}$$

$$f_7 = \frac{1}{2}x(x+1) \tag{18}$$

Therefore, the two unknowns of displacement w and shear force V of the problem will be determined through 7 parameters including the displacement, first derivative of displacement, and shear force at the nodes of the element. The 7 parameters can be written in the following matrix form:

$$\{X\} = [w_1 \quad \theta_1 \quad w_2 \quad \theta_2 \quad V_1 \quad V_2 \quad V_3]^T \tag{19}$$

3.3 Formation of Element Stiffness Matrix, Load Matrix, and Overall Stiffness Matrix

The displacement w can be written in the following matrix form:

$$w = [f_1 \quad f_2 \quad f_3 \quad f_4 \quad 0 \quad 0 \quad 0]\{X\}$$

or, $w = [w]\{X\}$ (20)

The shear force V can be written in the following matrix form:

$$V = [0 \quad 0 \quad 0 \quad 0 \quad f_5 \quad f_6 \quad f_7]\{X\}$$

or, $V = \{V\}\{X\}$ (21)

The quadratic derivative of the displacement w:

$$w'' = [f''_1 \quad f''_2 \quad f''_3 \quad f''_4 \quad 0 \quad 0 \quad 0]\{X\}$$

or, $w'' = [w'']\{X\}$ (22)

The first derivative of the shear force V:

$$V' = [0 \quad 0 \quad 0 \quad 0 \quad f'_5 \quad f'_6 \quad f'_7]\{X\}$$

or, $V' = [V']\{X\}$ (23)

Bending strain χ:

$$\chi = \frac{M}{EI} = \left[-\frac{d^2y}{dx^2} + \frac{d}{dx}\left(\frac{KV}{GA}\right)\right] \tag{24}$$

$$\chi = [-f''_1 \quad -f''_2 \quad -f''_3 \quad -f''_4 \quad +f'_5 \quad +f'_6 \quad +f'_7]\{X\}$$

or, $\chi = [\chi]\{X\}$ (25)

Shear strain γ:

$$\gamma = \frac{\alpha V}{GA} = \frac{\alpha}{GA}[V]\{X\} \tag{26}$$

The bending moment M can be written in the following matrix form:

$$M = EI.\chi = EI[\chi]\{X\} \tag{27}$$

The displacement is the cubic function, so the load acting on the beam must be the concentrated force located at the nodes of the element. The load P can be written in the following matrix form:

$$\{P\} = [P_1 \quad 0 \quad P_2 \quad 0 \quad 0 \quad 0 \quad 0]^T \tag{28}$$

Applying the virtual work principle, the equation of the beam problem considering transverse shear strain is:

$$\frac{\Delta x}{2}\left(\int_{-1}^{1} M.\delta\left[[\chi].\{X\}\right]dx+\int_{-1}^{1} V.\delta\left[\frac{\alpha}{GA}[V]\{X\}\right]dx\right)-$$
$$-\{P\}\delta\left[[w]\{X\}\right]=0 \tag{29}$$

where Δx is the actual length of the element.

Equation (29) is equivalent to

$$\frac{\Delta x}{2}\left(\int_{-1}^{1} EI.[\chi]^T.[\chi]dx+\frac{\alpha}{GA}\int_{-1}^{1}[V]^T.[V]dx\right)\{X\}-$$
$$-\{P\}=0$$

or, $[K_e]\{X\}=\{P\}$ (30)

where $[K_e]$ is the 7×7 element stiffness matrix.

Each element has 7 parameters of the vector $[X]$ to be determined. When there are n elements in the general case, there will be 7n unknowns. When matching elements, at the nodes, the displacement is usually continuous; at the nodes where there is no force, the first derivative of displacement and shear force is also constant, so the number of the actual unknowns will be less than 7n. When the system has n elements, the results of the internal force and the node strain of the entire bending beam are equal to the inverse of the overall stiffness matrix of the system multiplied by the load vector $K(7n\times 7n)$.

A Matlab program has been written to solve the basic bending beam problems and its use will be demonstrated below.

4. NUMERICAL EXAMPLES

4.1 Example 1

The simply supported beam subjected to uniformly distributed load is shown in Fig.6.

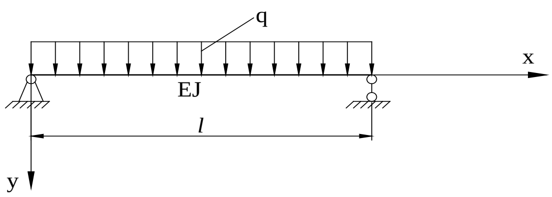

Fig.6 Simply supported beam subjected to uniformly distributed load

Calculation data: The simply supported beam of reinforced concrete is subjected to uniformly distributed load q = 10kN/m. The length of span $l=10m$. The cross-section dimensions of the beam are $b=0.5m,\ h=l/4=2.5m$. The elastic modulus $E=30000MPa$, Poisson's ratio $\nu=0.2$. Dividing the beam into 16 elements, the length of each element $\Delta x=0.625m$.

The results of calculation of the displacement, moment, shear force are shown in Fig.6a, Fig.6b, and Fig.6c.

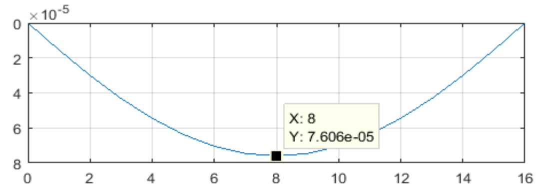

Fig.6a Displacement of simply supported beam subjected to uniformly distributed load

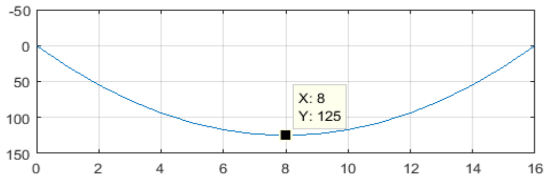

Fig.6b Moment of simply supported beam subjected to uniformly distributed load

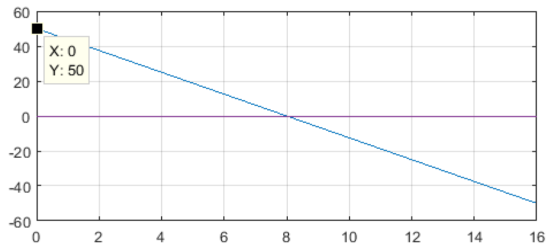

Fig.6c Shear force of simply supported beam subjected to uniformly distributed load

At $x=0$ and $x=l$, we have:

The displacement and moment of the beam are zero;

The shear forces at the ends of the beam: $V=\pm 50kN=\pm\frac{ql}{2}$.

At $x=l/2$, we have:

The maximum deflection at the middle of the beam:

$$w_{\max}=7.6\times10^{-5}m>\frac{5ql^4}{384EJ}=6.7\times10^{-5}m$$

The maximum bending moment at the middle of the beam: $M_{\max}=125kNm=\frac{ql^2}{8}$.

To compare the results of calculation to Professor Wilson's solutions, the author used Etabs software to calculate for the above problem (the software of which Professor Wilson is one of the authors). The results of calculation of the displacement, moment, and shear force are shown in Fig.6d, Fig.6e, and Fig.6f.

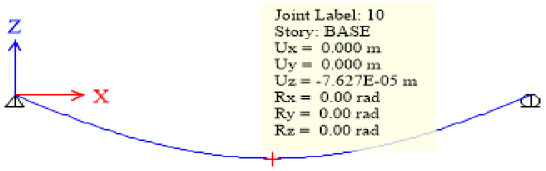

Fig.6d Displacement of simply supported beam subjected to uniformly distributed load

Fig.6e Moment of simply supported beam subjected to uniformly distributed load

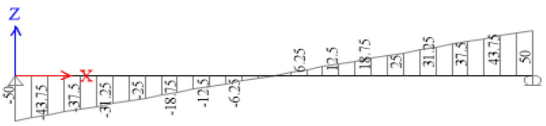

Fig.6f Shear force of simply supported beam subjected to uniformly distributed load

The simply supported beam subjected to uniformly distributed load, moment and shear force does not change when considering the transverse shear strain so the transverse shear strain is not considered.

The maximum deflection at the middle of the beam $w_{\max}$ is greater than that in the case of not considering the transverse shear strain.

The results of calculation are correct by the results on the Etabs software. On the other hand, these results also coincide with the results calculated according to the analytical method of Vu Thanh Thuy [4].

$$\left(w_{\max} = \frac{5ql^4}{384EJ} + \frac{ql^4}{40EJ}\left(\frac{h}{l}\right)^2 (1+\mu) = 7.6\times10^{-5} m \right)$$

When reducing the height of the beam (i.e. the ratio h/l is getting smaller, assuming $h/l = 1/10$), the displacement of the beam is shown in Fig.6g.

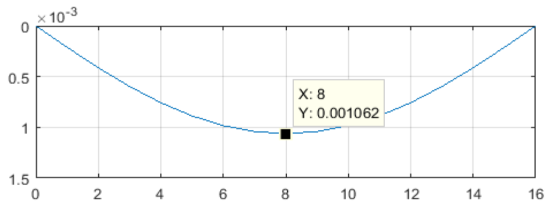

Fig.6g Displacement of simply supported beam subjected to uniformly distributed load

We see $w_{\max} = 0.001062m \approx \dfrac{5ql^4}{384EJ} = 0.001042m$.

It can be observed that the result of calculation according to beam theory considering the transverse shear strain has converged on the beam case without considering the transverse shear strain (without shear locking).

4.2 Example 2

The cantilever and simply supported beam subjected to uniformly distributed load are shown in Fig.7.

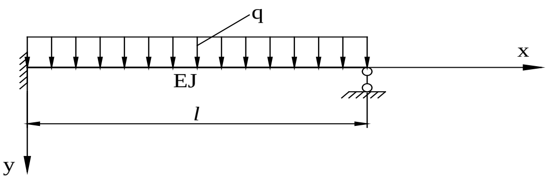

Fig.7 Cantilever and simply supported beam subjected to uniformly distributed load

Calculation data: The cantilever and simply supported beam of reinforced concrete are subjected to uniformly distributed load q = 10kN/m. The length of span $l = 10m$. The cross-section dimensions of the beam are $b = 0.5m$, $h = l/4 = 2.5m$. Elastic modulus $E = 30000MPa$, Poisson's ratio $\nu = 0.2$. Dividing the beam into 16 elements, the length of each element $\Delta x = 0.625m$.

The results of calculation of the displacement, moment, and shear force are shown in Fig.7a, Fig.7b, and Fig.7c.

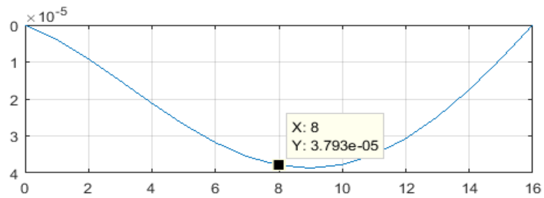

Fig.7a Displacement of cantilever and simply supported beam subjected to uniformly distributed load

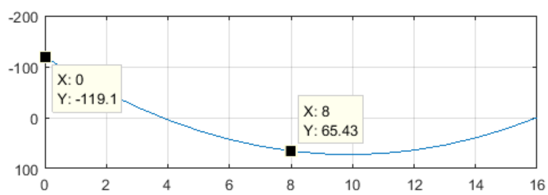

Fig.7b Moment of cantilever and simply supported beam subjected to uniformly distributed load

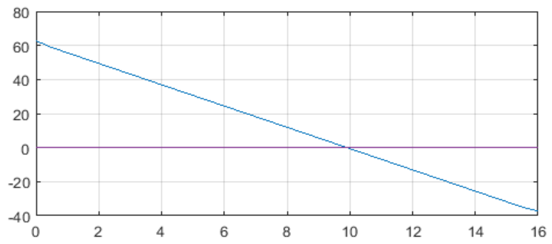

Fig.7c Shear force of cantilever and simply supported beam subjected to uniformly distributed load

At $x = 0$, we have:

The displacement of the beam is zero;

The moment at the cross-section next to the fixed support: $M = 119.1kNm < \dfrac{ql^2}{8} = 125kNm$;

The shear forces at the cross-section next to the fixed support: $V = 61.96kN < \dfrac{5ql}{8} = 62.5kN$.

At $x = l$, we have:

The displacement and moment of the beam are zero;

The shear forces at the cross-section next to the roller support: $V = 38.04kN > \dfrac{3ql}{8} = 37.5kN$.

At $x = l/2$, we have:

The deflection at the middle of the beam: $w_{\max} = 3.8 \times 10^{-5} m > \dfrac{ql^4}{192EJ} = 2.7 \times 10^{-5} m$

The bending moment at the middle of the beam: $M = 65.43kNm > \dfrac{ql^2}{16} = 62.5kNm$.

The results of calculation of the displacement, moment, and shear force by Etabs software are shown in Fig.7d, Fig.7e, and Fig.7f.

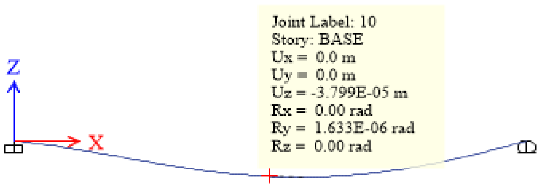

Fig.7d Displacement of cantilever and simply supported beam subjected to uniformly distributed load

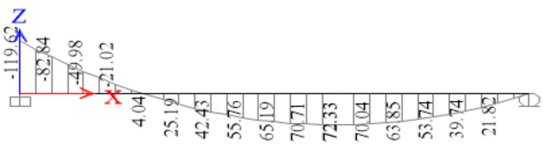

Fig.7e Moment of cantilever and simply supported beam subjected to uniformly distributed load

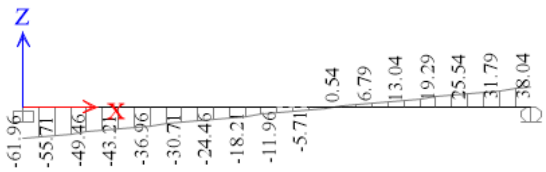

Fig.7f Shear force of cantilever and simply supported beam subjected to uniformly distributed load

The results of calculation are correct by the results on the Etabs software and the results are calculated according to the analytical method of Vu Thanh Thuy [4].

The moment at the cross-section next to the fixed support:

$$M = \frac{ql^2}{8} + \frac{3ql^2}{8}\left(\frac{\left(\frac{h}{l}\right)^2(1+\mu)}{5+3\left(\frac{h}{l}\right)^2(1+\mu)}\right) = 119.61kNm;$$

The shear forces at the cross-section next to the fixed support:

$$V = \frac{5ql}{8} - \frac{3ql}{8}\left(\frac{\left(\frac{h}{l}\right)^2(1+\mu)}{5+3\left(\frac{h}{l}\right)^2(1+\mu)}\right) = 61.96kN;$$

The shear forces at the cross-section next to the roller support:

$$V = \frac{3ql}{8} + \frac{3ql}{8}\left(\frac{\left(\frac{h}{l}\right)^2(1+\mu)}{5+3\left(\frac{h}{l}\right)^2(1+\mu)}\right) = 38.04kN.$$

The deflection at the middle of the beam:

$$w = \frac{ql^4}{192EJ}\left(1 + \frac{3}{10}.\frac{95+48\left(\frac{h}{l}\right)^2(1+\mu)}{5+3\left(\frac{h}{l}\right)^2(1+\mu)}\left(\frac{h}{l}\right)^2(1+\mu)\right)$$

$$= 3.8 \times 10^{-5} m$$

When considering the transverse shear strain, the moment and shear force at the cross-section next to the fixed support are reduced compared to not considering the transverse shear strain. Meanwhile, the moment at the middle of the beam increased by an amount equal to the amount it decreased at the fixed support; the shear force at the roller support is increased by an amount exactly equal to the amount it decreased at the fixed support. Clearly, there are redistributions of internal force in the beam when considering the transverse shear strain.

The deflection at the middle of the beam w is greater than that in the case of not considering transverse shear strain.

When reducing the height of the beam (i.e. the ratio h/l is getting smaller, assuming $h/l = 1/10$), the displacement of the beam is shown in Fig.7g.

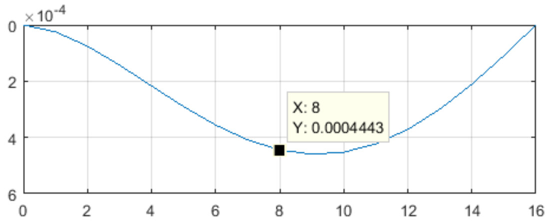

Fig.7g Displacement of cantilever and simply supported beam subjected to uniformly distributed load

We see $w = 0.00044m \approx \dfrac{ql^4}{192EJ} = 0.00042m$.

In addition, the problems of the fixed end beam subjected to uniformly distributed load, the fixed end beam subjected to concentrated load, the cantilever beam subjected to uniformly distributed load, and the cantilever beam subjected to concentrated load have also been realized in Matlab code. The results of calculation are correct by the results in the Etabs software and the results are calculated according to the analytical method of Vu Thanh Thuy. It can be observed that the improved method has achieved convergence and avoids the issue of shear locking.

5. CONCLUSION

An improved method of calculating the beam considering transverse shear strain is used to overcome the theoretical limitations of previous authors. The problem is addressed by using the finite element method, which achieves convergence and avoids the issue of shear locking.

According to the author's improved method, the results of calculation are correct by the results in the Etabs software and the results are calculated according to the analytical method of Vu Thanh Thuy.

By solving the bending beam problems, the effects of the transverse shear strain on the internal force, displacement and deformation in bending beams can be seen through some of the basic problems. The amount of this change depends on the connected condition, Poisson's ratio μ, and the ratio h/l. In particular, considering the transverse shear strain, we get a redistribution of the internal force in the beam (the cantilever and simply supported beam subjected to uniformly distributed load).

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