



A KINEMATIC ALGORITHM FOR PLASTIC ANALYSIS OF BEAM STRUCTURES BY DIRECT METHOD

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ABSTRACT: A kinematic-based direct method is proposed for evaluating limit and shakedown loads of beam structures subjected to proportional and cyclic loading. Conventional incremental elastoplastic analyses require detailed knowledge of the entire loading history and may involve high computational cost, which limits their application to complex structures. In contrast, direct methods allow the determination of critical load levels without tracing the full loading path. In this work, an upper bound formulation for limit and shakedown analysis of beam structures is developed using admissible plastic generalized strain fields within a kinematic framework. The resulting problem is formulated as a nonlinear optimization problem with compatibility and normalization constraints. A combined penalty–Lagrange technique is employed, and the corresponding Karush–Kuhn–Tucker conditions are solved using Newton iterations. The formulation is implemented within a finite element framework using the von Mises yield criterion. Several numerical examples involving statically indeterminate beams subjected to concentrated loads under proportional and cyclic loading are presented. The results show good agreement with analytical solutions and demonstrate fast and stable convergence of the proposed algorithm. These findings indicate that the proposed method provides an efficient and reliable tool for limit and shakedown analysis of beam structures.

Keywords: Limit analysis, Shakedown analysis, Plastic analysis of structures, Non-linear programming, Finite element method.

1. INTRODUCTION

Limit and shakedown analysis play a crucial role in the safety assessment and optimal design of structural systems subjected to severe, repeated, or variable loading. In many engineering applications, such as bridges, industrial buildings, offshore structures, and energy-related infrastructure, structural components may experience cyclic or non-monotonic loads that can lead to plastic deformation, accumulation of residual strains, and eventual collapse [1-4,9-10]. Traditional elastic design methods are often conservative because they do not account for the redistribution of internal forces after yielding. Plastic analysis, and in particular limit analysis, enables engineers to exploit the reserve strength of structures and to estimate their true load-carrying capacity more realistically.

Classical approaches for determining ultimate load levels are commonly based on incremental elastoplastic analyses. In these methods, the loading process is divided into small steps and the evolution of stresses and deformations is traced until collapse occurs [21]. Although conceptually straightforward, such approaches require detailed information about

the entire loading history and involve high computational cost, especially for cyclic loading conditions. Furthermore, numerical difficulties such as stress singularities, convergence problems, and path-dependence effects may arise in elastic–plastic finite element simulations.

To overcome these limitations, direct methods for limit and shakedown analysis have been developed [4-8,11-13]. These methods reformulate the structural problem as a mathematical optimization problem and allow the direct determination of critical load multipliers without simulating the full loading path [1,3,6-8,13-14,19-20,24-28]. Two main classes of direct methods are commonly distinguished: static (lower bound) and kinematic (upper bound) approaches. Among them, kinematic methods are particularly attractive because of their clear physical interpretation and their ability to provide guaranteed upper bounds for collapse and shakedown loads.

In recent years, research on limit and shakedown analysis has increasingly emphasized algorithmic efficiency and numerical robustness. Several works have proposed numerical procedures that improve computational performance for practical structures,

including railway slab track foundations under dynamic loading [29], and chance-constrained programming frameworks for shakedown analysis under stochastic material properties [30]. Notably, optimization-based strategies involving nested algorithms have been developed to maximize shakedown capacity within topology optimization contexts, as demonstrated by Li and Cheng [31], who derived concise analytical sensitivity expressions and implemented a primal–dual shakedown algorithm for truss structures. In parallel, materials engineering research has explored deformation mechanisms under complex loading conditions, including shakedown behavior in alloys subjected to long-term fatigue–creep interactions [32]. Recent studies in the *International Journal of GEOMATE* have reported on responses of cohesive soils under cyclic loading [33], underscoring the relevance of shakedown concepts across structural domains. Additionally, advanced shakedown formulations under coupled cyclic mechanical–thermal loads have been presented in related journals [34], reinforcing the breadth of recent developments in both theoretical and application-oriented frameworks.

However, most existing approaches are either focused on specific material models, optimization frameworks, or particular discretization techniques, and often require complex formulations or high computational effort. In particular, a systematic and unified kinematic formulation for upper bound limit and shakedown analysis of beam structures, combined with an efficient and robust numerical algorithm, is still limited. Existing methods rarely address both limit and shakedown analysis within a single kinematic framework while maintaining simplicity, fast convergence, and independence from load history.

Therefore, there remains a clear need for an efficient kinematic algorithm based on the direct method that can accurately evaluate upper bound limit and shakedown loads of beam structures under proportional and cyclic loading. Addressing this gap forms the main motivation of the present study.

The main contribution of this study is the development of a kinematic-based algorithm for limit and shakedown analysis of beam structures within a continuum finite element framework. The proposed formulation combines penalty and Lagrange multiplier techniques to transform the constrained nonlinear optimization problem into a form that can be efficiently solved using Newton iterations. Compared with existing direct methods, the proposed approach provides a stable numerical procedure with fast convergence and straightforward implementation in finite element analysis.

The remainder of this paper is organized as follows. Section 2 presents the research significance of the study. Section 3 introduces the theoretical formulation of the upper bound method for limit and shakedown analysis. Section 4 describes the numerical algorithm used to solve the resulting optimization problem. Section 5 presents several numerical examples to demonstrate the accuracy and efficiency of the proposed approach. Finally, the main conclusions of the study are summarized in Section 6.

2. RESEARCH SIGNIFICANCE

This study presents a novel kinematic algorithm for upper bound limit and shakedown analysis of beam structures within a direct method framework. The originality of the work lies in the systematic formulation of admissible plastic generalized strain cycles combined with an efficient nonlinear programming strategy solved by Newton iterations. Compared with conventional incremental elastoplastic analyses and existing direct approaches, the proposed method avoids load-history dependence and significantly reduces computational effort while maintaining high accuracy. In addition, the formulation is implemented within a two-dimensional continuum finite element framework, allowing beam problems to be analyzed using plane stress elements and enabling a more general representation of stress and strain fields. The developed algorithm therefore provides a robust and practical computational tool for evaluating structural safety and load-carrying capacity of beam systems subjected to cyclic and variable loading.

Table 1. Comparison of some selected numerical approaches.

Author	Formulation	Characteristic
Christiansen [6]	Upper bound limit analysis	Accurate formulation but relatively high computational cost
Lyamin& Sloan [13]	Lower bound limit analysis	Robust solution procedure but complex implementation
Garcea et al. [14]	FEM shakedown analysis	Suitable for large-scale structures but requires complex formulations
Gaudrat [7]	Plastic limit analysis	Efficient numerical solution but focused mainly on limit analysis
Li&Cheng[31]	Shakedown-based topology optimization	Designed for topology optimization of truss structures rather than general beam analysis
Present study	Upper bound limit and shakedown analysis	Fast convergence, unified framework for limit and shakedown analysis,guilstraightforward FEM implementation

As summarized in Table 1, several numerical approaches have been developed for limit and

shakedown analysis of structures. However, many existing methods rely on complex optimization procedures or focus primarily on either limit analysis or shakedown analysis. In contrast, the proposed formulation combines a kinematic upper bound framework with a penalty–Lagrange numerical strategy solved by Newton iterations. This approach enables the unified treatment of limit and shakedown problems while maintaining computational efficiency and straightforward implementation within a finite element framework.

3. UPPER BOUND FORMULATIONS FOR LIMIT AND SHAKEDOWN ANALYSIS OF BEAM STRUCTURES

3.1 Continuum Formulation

Using a plastic strain field to formulate a shakedown criterion, the kinematic shakedown theorem is the counterpart of the static one. The theorem was given by Koiter [16] and some of its applications in analysis of incremental collapse were derived by Gokhfeld and Sawczuk. Same as proposed by Koiter for plastic strain field, we introduce here an admissible cycle of plastic generalized strain field $\Delta \epsilon_p$. The plastic generalized strain rate $\dot{\epsilon}^p$ may not necessarily be compatible at each instant during the period T but the plastic generalized strain accumulation over the period:

$$\Delta \epsilon_p = \int_0^T \dot{\epsilon}_p dt \quad (1)$$

is required to be compatible such that

$$\Delta \epsilon_p = \nabla [(\Delta \mathbf{u})]_{\text{sym}} \quad (2)$$

$$\Delta \mathbf{u} = 0 \quad \text{on } S_u$$

$$\text{and } \int_V \int_0^T \sigma_e : \dot{\epsilon}_p dt dV > 0 \quad (3)$$

where $\mathbf{u}(\mathbf{x})$ denotes the displacement vector in the structure.

Kinematic theorem is stated as follows:

Shakedown may happen if the following inequality is satisfied

$$\int_V \int_0^T \sigma_e(\mathbf{x}, t) : \dot{\epsilon}_p dV dt \leq \int_V \int_0^T D^p(\dot{\epsilon}_p) dV dt \quad (4)$$

Shakedown cannot happen when the following inequality holds

$$\int_V \int_0^T \sigma_e(\mathbf{x}, t) : \dot{\epsilon}_p dV dt > \int_V \int_0^T D^p(\dot{\epsilon}_p) dV dt \quad (5)$$

Based on the kinematic theorem, an upper bound of the shakedown limit load multiplier α^+ can be computed. The shakedown problem can be seen as a mathematical minimization problem in nonlinear programming.

$$\begin{aligned} \alpha^+ &= \min \int_V \int_0^T D^p(\dot{\epsilon}_p) dV dt \\ \text{s.t.: } &\begin{cases} \Delta \epsilon_p = \int_0^T \dot{\epsilon}_p dt \\ \Delta \epsilon_p = \nabla [(\Delta \mathbf{u})]_{\text{sym}} & \text{in } V \\ \Delta \mathbf{u} = 0 & \text{on } S_u \\ \int_V \int_0^T \sigma_e(\mathbf{x}, \hat{P}_k) : \dot{\epsilon}_p dV dt = 1 \end{cases} \end{aligned} \quad (6)$$

3.2 FEM Shakedown Analysis Based on Kinematic Approach

In order to solve the shakedown problem numerically, the continuum formulation presented above is discretized using the finite element method. The plastic strain rate field and displacement field are approximated within each finite element, and the global optimization problem is obtained by assembling the contributions from all Gauss integration points. Based on this discretization, the upper bound shakedown problem can be written in the following finite element form.

If the von Mises criteria is used the discretized formulation of the upper bound shakedown problem is:

$$\begin{aligned} \alpha^+ &= \min \sum_{k=1}^m \sum_{i=1}^{NG} \sqrt{\frac{2}{3}} \sigma_y \sqrt{\dot{\epsilon}_{ik}^T \mathbf{D} \dot{\epsilon}_{ik}} \\ \text{s.t.: } &\begin{cases} \sum_{k=1}^m \dot{\epsilon}_{ik} - \mathbf{B}_i \dot{\mathbf{u}} = \mathbf{0} \quad \forall i = \overline{1, NG} \\ \sum_{k=1}^m \sum_{i=1}^{NG} w_i \dot{\epsilon}_{ik}^T \sigma_{ik}^e - 1 = 0 \end{cases} \end{aligned} \quad (7)$$

Where $\dot{\epsilon}_{ik}$ is the strain rate vector at Gauss point i , corresponding to the k^{th} load vertex. σ_{ik}^e is the fictitious elastic stress vector at Gauss point i corresponding to the k^{th} vertex. $\dot{\mathbf{u}}$ is the displacement rate vector. $\mathbf{B}_i$ is the deformation matrix at Gauss point $\mathbf{x}_i$. w_i is the weight factor of the Gauss point.

Problems (7) is an optimization problem, we can use Lagrange multiplier method and penalty method to convert the problem with constraints into an unconstrained programming problem. After that we solve Karush-Kuhn-Tucker conditions of the unconstrained programming problem to obtain the optimal solution [24].

4. ALGORITHM FOR SOLUTIONS

4.1 Karush–Kuhn–Tucker conditions

Some new variables are introduced for easy computation:

$\dot{\mathbf{x}}_i = \mathbf{w}_i \mathbf{D}^{1/2} \dot{\mathbf{e}}_i$, $\mathbf{s}_i = \mathbf{D}^{-1/2} \boldsymbol{\sigma}_i^E$, $\mathbf{Q} = \mathbf{w}_i \mathbf{D}^{1/2} \mathbf{B}_i$, $\mathbf{D}^{1/2}$ and $\mathbf{D}^{-1/2}$ are symmetric matrixes which satisfy $(\mathbf{D}^{1/2})^{-1} = \mathbf{D}^{-1/2}$ and $\mathbf{D} = (\mathbf{D}^{1/2} \mathbf{D}^{1/2})$.

Substituting these variables into (7) we have

$$\alpha^+ = \min \sum_{i=1}^{NG} \sqrt{\frac{2}{3}} \sigma_Y \sqrt{\dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_i + \varepsilon_0^2}$$

$$\text{s.t.} \begin{cases} \dot{\mathbf{x}}_i = \mathbf{Q}_i \dot{\mathbf{u}} \\ \sum_{i=1}^{NG} \dot{\mathbf{x}}_i^T \mathbf{s}_i = 1 \end{cases} \quad (8)$$

The penalty method and the Lagrange method are used simultaneously to convert the problem (8) into an unconstrained programming:

$$\alpha = \min Y \quad (9)$$

Where

$$Y = \sum_{i=1}^{NG} \sqrt{\frac{2}{3}} \sigma_Y \sqrt{\dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_i + \varepsilon_0^2} + \frac{C}{2} (\dot{\mathbf{x}}_i - \mathbf{Q}_i \dot{\mathbf{u}})^T (\dot{\mathbf{x}}_i - \mathbf{Q}_i \dot{\mathbf{u}}) + \lambda \left[\sum_{i=1}^{NG} \dot{\mathbf{x}}_i^T \mathbf{s}_i - 1 \right] \quad (10)$$

here C is a penalty parameter, $C \gg 1$, λ is the Lagrange multiplier.

The Karush–Kuhn–Tucker conditions (KKT optimality conditions) gives the equations to obtain optimal solution

$$\begin{cases} \frac{\partial Y}{\partial \dot{\mathbf{x}}_i} = \left(\sigma_{Yi} \frac{\dot{\mathbf{x}}_i}{\sqrt{\dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_i + \varepsilon^2}} \right) + c (\dot{\mathbf{x}}_i - \mathbf{Q}_i \dot{\mathbf{u}}) + \lambda \mathbf{s}_i = 0 & (a) \\ \frac{\partial Y}{\partial \dot{\mathbf{u}}} = -c \mathbf{Q}_i^T (\dot{\mathbf{x}}_i - \mathbf{Q}_i \dot{\mathbf{u}}) = 0 & (b) \\ \frac{\partial Y}{\partial \lambda} = \sum_{i=1}^{NG} \dot{\mathbf{x}}_i^T \mathbf{s}_i - 1 = 0 & (c) \end{cases} \quad (11)$$

The Newton method is applied to solve the system (11) of nonlinear equations. We get:

$$d\dot{\mathbf{u}} = d\dot{\mathbf{u}}_1 + (\lambda + d\lambda) d\dot{\mathbf{u}}_2 \quad (12)$$

$$d\dot{\mathbf{x}}_i = (d\dot{\mathbf{x}}_i)_1 + (\lambda + d\lambda) (d\dot{\mathbf{x}}_i)_2 \quad (13)$$

$$\lambda + d\lambda = \frac{1 - \sum_{k=1}^{NG} (\dot{\mathbf{x}}_i^T \mathbf{s}_i) - \sum_{i=1}^{NG} (\mathbf{s}_i^T) (d\dot{\mathbf{x}}_i)_1}{\sum_{i=1}^{NG} (\mathbf{s}_i^T) (d\dot{\mathbf{x}}_i)_2} \quad (14)$$

in which:

$$\begin{aligned} d\dot{\mathbf{u}}_1 &= -\dot{\mathbf{u}} & d\dot{\mathbf{u}}_2 &= \mathbf{S}^{-1} \mathbf{f}_{u2} & (d\dot{\mathbf{x}}_i)_1 &= -\dot{\mathbf{x}}_i \\ (d\dot{\mathbf{x}}_i)_2 &= \mathbf{H}_i^{-1} \sqrt{\dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_i + \varepsilon_0^2} \mathbf{s}_i \\ &+ \sqrt{\dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_i + \varepsilon_0^2} \mathbf{H}_i^{-1} \mathbf{E}_i^{-1} \hat{\mathbf{Q}}_i d\dot{\mathbf{u}}_2 \\ &- \sqrt{\dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_i + \varepsilon_0^2} \mathbf{H}_i^{-1} \mathbf{E}_i^{-1} \mathbf{H}_i^{-1} \sqrt{\dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_i + \varepsilon_0^2} \mathbf{s}_i \\ \mathbf{H}_i &\approx \left(\sigma_{Yi} \sqrt{\dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_i + \varepsilon^2} \right) \mathbf{I}_i \\ \mathbf{S} &= \sum_{i=1}^{NG} \hat{\mathbf{Q}}_i^T \mathbf{E}_i^{-1} \hat{\mathbf{Q}}_i \\ \mathbf{f}_{u2} &= \sum_{i=1}^{NG} \hat{\mathbf{Q}}_i \mathbf{E}_i^{-1} \mathbf{H}_i^{-1} \sqrt{\dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_i + \varepsilon_0^2} \mathbf{s}_i \\ \mathbf{E}_i &= \left(\frac{\mathbf{I}_i}{C} + \sqrt{\dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_i + \varepsilon_0^2} \mathbf{H}_i^{-1} \right) \end{aligned} \quad (15)$$

• **Step 2:** Calculate $\mathbf{S}$, $\mathbf{f}_{u2}$ from (12) at current values of $\dot{\mathbf{u}}$ and $\dot{\mathbf{x}}$

• **Step 3:** Calculate $(\lambda + d\lambda)$, $d\dot{\mathbf{u}}$, $d\dot{\mathbf{x}}_{ik}$ from equations (12-14).

• **Step 4:** Update displacement, strain rate and λ as

$$\begin{aligned} \dot{\mathbf{u}} &= \dot{\mathbf{u}} + d\dot{\mathbf{u}} \\ \dot{\mathbf{x}}_{ik} &= \dot{\mathbf{x}}_{ik} + d\dot{\mathbf{x}}_{ik} \\ \lambda &= \lambda + d\lambda \end{aligned}$$

• **Step 5:** Check convergence criteria: if they are all satisfied go to step 6 otherwise repeat steps 2, 3 and 4.

• **Step 6:** Stop

5. NUMERICAL EXAMPLE

5.1 Limit analysis of beam subjected to a concentrated force

Consider the beam in [28] subjected to a concentrated force as shown in Figure 3. Let us compute the limit of a single load acting on the beam. The length of the beam is $L = 4$ m. The beam has a rectangular cross-section with the plastic moment $M_p = 4$ kNm.

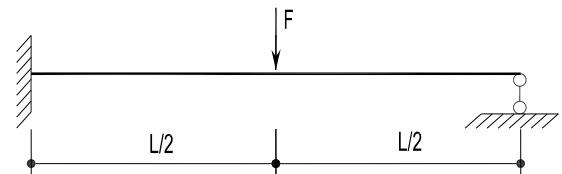


Fig. 1: Statically indeterminate beam subjected to a point force

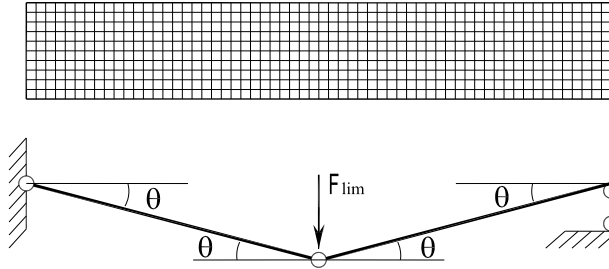


Fig. 2. FE mesh with 400 elements and collapse mechanism of a beam

Analytical solution: At limit state two plastic hinges form as shown in Figure 6, plastic dissipation energy is equal to external work. This leads to the following equation:

$$\dot{W}_{in} = \dot{W}_{ext}, \quad (16)$$

in which:

$$\dot{W}_{in} = M_p \theta + M_p \cdot 2\theta = 3M_p \theta; \quad \dot{W}_{ext} = (FL/2)\theta.$$

From equation (16) we have $F_{lim} = 6M_p/L$.

Substituting the input numerical data: $M_p = 4 \text{ kNm}$, $L = 4 \text{ m} \rightarrow F_{lim} = 6 \text{ kN}$.

FEM solutions: The beam is modeled by 400 rectangular finite elements, the algorithm, which is mentioned above, gives the limit load $F_{lim} = 6.45 \text{ kN}$. If we increase the number of elements to 600 and 800 then the results are 6.38 kN and 6.25 kN, respectively. The results are listed in Table 1. The numerical results show that the computed limit load converges toward the analytical solution as the number of finite elements increases. This behavior confirms the stability and accuracy of the proposed algorithm.

The beam is statically indeterminate of degree 1. The absolute maximum bending moment in the elastic analysis is $|M(x=0)| = 3FL/16$ and occurs at the clamped end [28]. For a beam with rectangular cross-section the elastic moment at first yield is $M_e = 2M_p/3$ so that the yield limit is $F_{el} = 16M_e/3L = 32M_p/9L$. With the given data $F_{el} = 3.556 \text{ kN}$, which is only 59.3% of the analytical plastic collapse limit F_{lim} and shows the great benefit of the increased load carrying capacity of plastic structural design by limit analysis compared to traditional elastic structural design. In contrast to the plastic limit analysis, the elastic FEM model has the difficulty that singular stress occurs at the point load as well as at both supports.

The statically indeterminate linear beam analysis is much more involved than the plastic limit analysis. The beam is statically determinate after the first plastic hinge has formed so that residual stresses

disappear and the limit load becomes also independent of the load history.

The beam model handles stress resultants (bending moments, shear forces) while plane stress FEM deals with strains and stresses directly. Point forces, which are a common feature in beam theory, create stress and displacement singularities when applied to a plane stress element in FEM. Singularities in plane stress FEM also occur at point supports and clamped end. Here the singularities cannot be ignored in elastic plane stress because they appear at the points of interest (points with maximum moments). Therefore, it is difficult to get meaningful yield limits from plane stress FEM. No singularities appear in the beam model. They also do not appear in plastic limit analysis. No shakedown is possible if singularities occur in the elastic analysis. This difficulty is related to the presence of stress singularities in elastic finite element solutions under concentrated loads or point constraints, which prevent the definition of a bounded elastic stress field required in shakedown analysis (see e.g. Koiter [16]; Hill [22]).

Table 1. Results of analytical and numerical solutions

Models	Limit load (kN)
Analytical (beam)	6
FEM - 400 elements	6.45
FEM - 600 elements	6.38
FEM - 800 elements	6.25

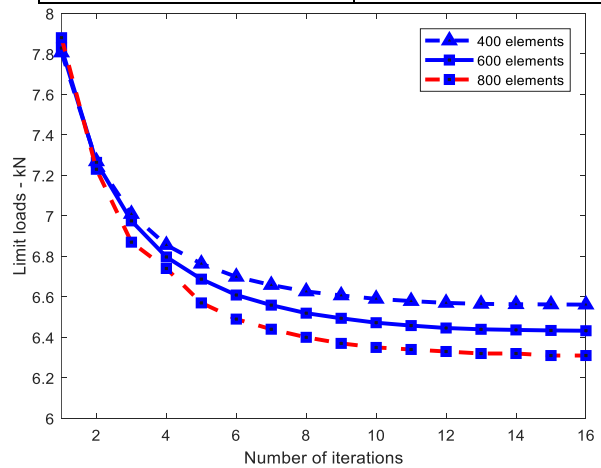


Fig. 3. Convergence of the limit loads

5.2 Shakedown analysis of a beam subjected to a point force

In this analysis, we compute the shakedown load of a beam, as shown in Figure 2, in two cases:

- F varies in the load domain $\mathcal{L} = [0, 4]$ (kN)
- F varies in the load domain $\mathcal{L} = [-4, 4]$ (kN)

The beam is modelled with 800 rectangular elements, the investigations show that for the first case, the shakedown load matches the limit load. For the second load domain, the force P varies in range from -4 kN to 4 kN, the shakedown load is 3.65 kN. These results are listed in Table II. The convergence of the shakedown load is shown in Figure 4.

Table 2. Shakedown loads in different load domains

Load domain (kN)	$[0, 4]$	$[-4, 4]$
Shakedown load (kN)	6.25	3.65

5.3 Shakedown analysis of beam subjected to two forces

In the third example, we consider a beam subjected to two concentrated forces F_1, F_2 which can vary in a load domain as shown in Figure 5. The limit analysis of this problem was investigated in [28] for the case $F_1 = F_2$ and the analytical solution $F_{lim} = 4M_p/L$ for the limit load was found by both lower bound and upper bound approaches.

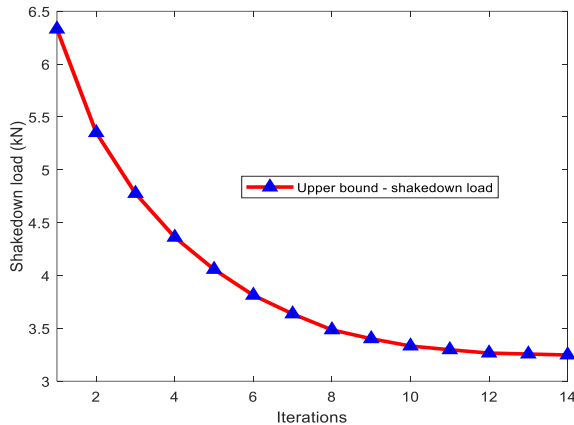


Fig. 4. Convergence of the shakedown load

The maximum bending moment for the elastic analysis of the given simply supported beam with two concentrated loads also occurs at the clamped end and is $M(x=0) = F_1 L/3$ for the case $F_1 = F_2$. For a beam with rectangular cross-section $M_e = 2M_p/3$. Thus $F_{el} = 3M_e/L = 2M_p/L$ or normalized with $4M_p/L$ the elastic limit of first yield is 0.5 analytically.

In the present analysis, the beam is modelled by 600 elements. Numerical results are listed in Table III which are normalized with $4M_p/L$. In the case F_1, F_2 vary independently in ranges the load domain $\mathcal{L} = [0, 1]P \times [0, 1]P$ the shakedown load matches the limit load at 1.033 .

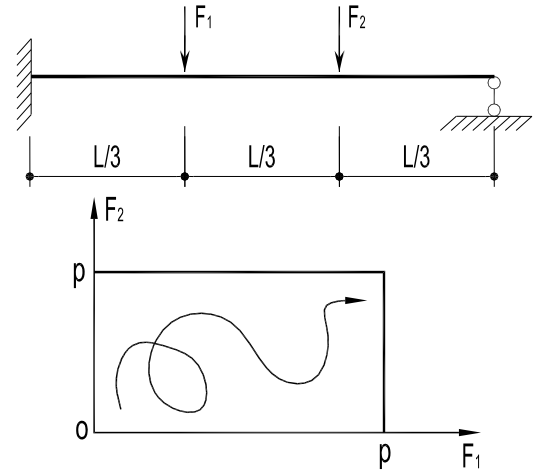


Fig. 5. Beam subjected to two forces with a load domain

In Table II, present results are compared with the analytical limit load in [28]. Figure 6 shows an interaction diagram. The interaction diagram illustrates the relationship between the two load parameters and identifies three distinct structural responses. In the elastic region, the applied loads remain below the elastic limit and no permanent deformation occurs.

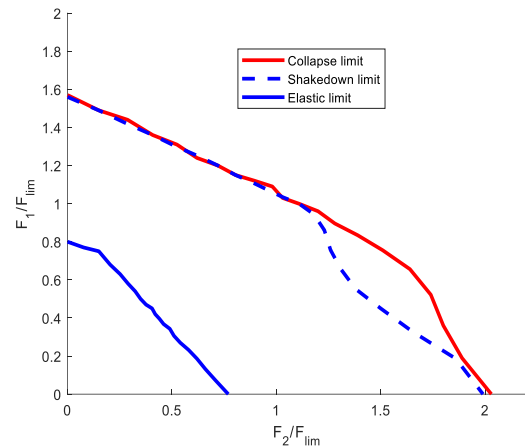


Fig. 6. Interaction diagram illustrating elastic, shakedown and collapse regions ($\times 4M_p/L$)

In the shakedown region, limited plastic deformation may occur during the initial loading cycles, but the structure eventually adapts and responds elastically to subsequent load cycles. When the load combination exceeds the shakedown boundary, incremental plastic deformation develops and the structure progresses toward collapse. Therefore, the diagram provides a clear graphical interpretation of the transition between elastic behavior, shakedown response, and collapse for different load combinations.

Table 3. Load limits normalized with $4M_p/L$

Method	Yield load	Limit load	Shakedown load
Analytical	0.5	1.0 [28]	-
FEM	0.43	1.034	1.033

6. CONCLUSIONS

Limit and shakedown analysis plays an important role in evaluating the load-carrying capacity and long-term safety of engineering structures subjected to variable loading conditions. In particular, structures exposed to cyclic or repeated loads may experience the accumulation of plastic deformation even when the applied loads remain within the elastoplastic domain. For this reason, efficient computational approaches for determining limit and shakedown load levels are of significant interest in structural analysis and design.

This study presented a kinematic-based direct method for the limit and shakedown analysis of beam structures. The proposed formulation is based on the upper bound theorem and employs admissible plastic generalized strain fields within a finite element framework. The resulting constrained nonlinear optimization problem was solved using a combined penalty and Lagrange multiplier technique, and the corresponding Karush–Kuhn–Tucker optimality conditions were efficiently handled using Newton iterations.

Several numerical examples involving statically indeterminate beam structures subjected to concentrated loads under proportional and cyclic loading were investigated. The numerical results showed good agreement with available analytical solutions and reference results, confirming the accuracy of the proposed approach. In addition, the algorithm exhibited stable numerical performance and rapid convergence, typically within a small number of iterations.

Overall, the proposed kinematic algorithm provides an efficient and reliable computational framework for plastic and shakedown analysis of beam structures. The method can be readily implemented within finite element procedures and has potential applications in practical structural design and safety assessment under variable loading conditions.

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